Local indicability and commutator subgroups of Artin groups

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Abstract

Artin groups (also known as Artin-Tits groups) are generalizations of Artin's braid groups. This paper concerns Artin groups of spherical type, that is, those whose corresponding Coxeter group is finite, as is the case for the braid groups. We compute presentations for the commutator subgroups of the irreducible spherical-type Artin groups, generalizing the work of Gorin and Lin [GL69] on the braid groups. Using these presentations we determine the local indicability of the irreducible spherical Artin groups (except for F_4 which at this time remains undetermined). We end with a discussion of the current state of the right-orderability of the spherical-type Artin groups.

1 Introduction

A number of recent discoveries regarding the Artin braid groups $\mathfrak{B}_n$ complete a rather interesting story about the orderability[1] of these groups. These discoveries were as follows.

In 1969, Gorin and Lin [GL69], by computing presentations for the commutator subgroups $\mathfrak{B}'_n$ of the braid groups $\mathfrak{B}_n$, showed that $\mathfrak{B}'_3$ is a free group of rank 2, $\mathfrak{B}'_4$ is the semidirect product of two free groups (each of rank 2), and $\mathfrak{B}'_n$ is finitely generated and perfect for $n \geq 5$. It follows from these results that $\mathfrak{B}_n$ is locally indicable[2] if and only if $n < 5$.

Neuwirth in 1974 [Neu74], observed $\mathfrak{B}_n$ is *not* bi-orderable if $n \geq 3$. Twenty years later, Dehornoy [Deh94] showed the braid groups are in fact right-orderable for all n. Furthermore, it has been observed, [RZ98], [KR03], that the subgroups $\mathcal{P}_n$ of pure braids are bi-orderable.

These orderings were fundamentally different, and it was natural to ask if there might be compatible orderings, that is a right-invariant ordering of $\mathfrak{B}_n$

[1] A group G is *right-orderable* if there exists a strict total ordering $<$ of its elements which is right-invariant: $g < h$ implies $gk < hk$ for all $g, h, k \in G$. If in addition $g < h$ implies $kg < kh$, the group is said to be *orderable*, or for emphasis, *bi-orderable*.

[2] A group G is *locally indicable* if for every nontrivial, finitely generated subgroup H of G there exists a *nontrivial* homomorphism $H \to \mathbb{Z}$.

which restricts to a bi-ordering of $\mathcal{P}_n$. This question was answered by Rhemtulla and Rolfsen [RR02] by exploiting the connection between local indicability and orderability. They showed that since the braid groups $\mathfrak{B}_n$ are not locally indicable for $n \geq 5$ a right-ordering on $\mathfrak{B}_n$ could not restrict to a bi-ordering on $\mathcal{P}_n$ (or on any subgroup of finite index).

This paper is concerned with investigating the extent to which of these results on the braid groups extend to other Artin groups, or at least those of spherical type (defined in the next section). In particular, we are concerned with determining the local indicability of the spherical Artin groups. Because the full details of the Gorin-Lin calculations do not seem to appear in the literature, we present a fairly comprehensive account of the calculation of commutator subgroups of the braid groups, which are the Artin groups of type A_n. These methods, essentially the Reidemeister-Schreier method plus a few tricks, are also used to calculate presentations of the commutator subgroups of the other spherical Artin groups.

In the next section we will define Coxeter graphs Γ, the corresponding Coxeter groups W_Γ and the Artin groups $\mathcal{A}_\Gamma$. The spherical Artin groups are classified according to types: $A_n (n \geq 1), B_n (n \geq 2), D_n (n \geq 4), E_6, E_7, E_8, F_4, H_3, H_4$ and $I_2(m), (m \geq 5)$. Our main results are summarized in the following theorem, where $\mathcal{A}'_\Gamma$ denotes the commutator subgroup. Recall that a perfect group is one which equals its own commutator subgroup; any homomorphism from a perfect group to an abelian group must be trivial.

Theorem 1.1 *The following commutator subgroups are finitely generated and perfect:*

1. $\mathcal{A}'_{A_n}$ *for* $n \geq 4$,
2. $\mathcal{A}'_{B_n}$ *for* $n \geq 5$,
3. $\mathcal{A}'_{D_n}$ *for* $n \geq 5$,
4. $\mathcal{A}'_{E_n}$ *for* $n = 6, 7, 8$,
5. $\mathcal{A}'_{H_n}$ *for* $n = 3, 4$.

Hence, the corresponding Artin groups are not locally indicable.

On the other hand, we show the remaining spherical-type Artin groups *are* locally indicable (excluding the type F_4 which at this time remains undetermined) .

In a final section we discuss the orderability of the spherical-type Artin groups. We show that to determine the orderability of the spherical-type Artin groups it is sufficient to consider the positive Artin monoid. Furthermore, we show that to prove *all* spherical-type Artin groups are right-orderable it would suffice to show the Artin group (or monoid) of type E_8 is right-orderable.

2 Coxeter and Artin groups

Let S be a finite set. A **Coxeter matrix** over S is a matrix $M = (m_{ss'})_{s,s' \in S}$ indexed by the elements of S and satisfying

(a) $m_{ss} = 1$ if $s \in S$,
(b) $m_{ss'} = m_{s's} \in \{2, \dots, \infty\}$ if $s, s' \in S$ and $s \neq s'$.

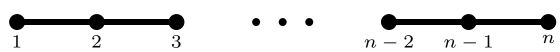
A Coxeter matrix $M = (m_{ss'})_{s,s' \in S}$ is usually represented by its **Coxeter graph** Γ. This is defined by the following data.

(a) S is the set of vertices of Γ.
(b) Two vertices $s, s' \in S$ are joined by an edge if $m_{ss'} \geq 3$.
(c) The edge joining two vertices $s, s' \in S$ is labelled by $m_{ss'}$ if $m_{ss'} \geq 4$.

The **Coxeter system** of type Γ (or M) is the pair (W, S) where W is the group having the presentation

$$W = \langle s \in S : (ss')^{m_{ss'}} = 1 \text{ if } m_{ss'} < \infty \rangle.$$

Example 2.1 *It is well known that the symmetric group on $(n+1)$-letters is the Coxeter group associated with the Coxeter graph;*



where vertex i corresponds to the transposition $(i\ i+1)$.

If (W, S) is a Coxeter system with Coxeter graph Γ (resp. Coxeter matrix M) then we say that Γ (resp. M) is of **spherical-type** if W is finite. If Γ is connected, then W is said to be **irreducible**. Coxeter [Cox34] classified all irreducible Coxeter groups which are finite, a result that plays a central role in the theory of Lie groups. We refer the reader to [Hum72] (see also Bourbaki [Bou72], [Bou02]) for further details on Coxeter groups, including a proof of the following.

Theorem 2.2 *The connected Coxeter graphs of spherical type are exactly those depicted in figure 1.*

The letter beside each of the graphs in figure 1 is called the **type** of the Coxeter graph; the subscript denotes the number of vertices. Recall example 2.1 shows the symmetric group on (n+1)-letters is a Coxeter group of type A_n.

Let M be a Coxeter matrix over S as described above, and let Γ be the corresponding Coxeter graph. Fix a set Σ in one-to-one correspondence with S.

We adopt the following notation, where $a, b \in \Sigma$:

$$\langle ab \rangle^q = \underbrace{aba\dots}_{\text{q factors}}$$

The **Artin system** of type Γ (or M) is the pair $(\mathcal{A}, \Sigma)$ where $\mathcal{A}$ is the group having presentation

$$\mathcal{A} = \langle a \in \Sigma : \langle ab \rangle^{m_{ab}} = \langle ba \rangle^{m_{ab}} \text{ if } m_{ab} < \infty \rangle.$$

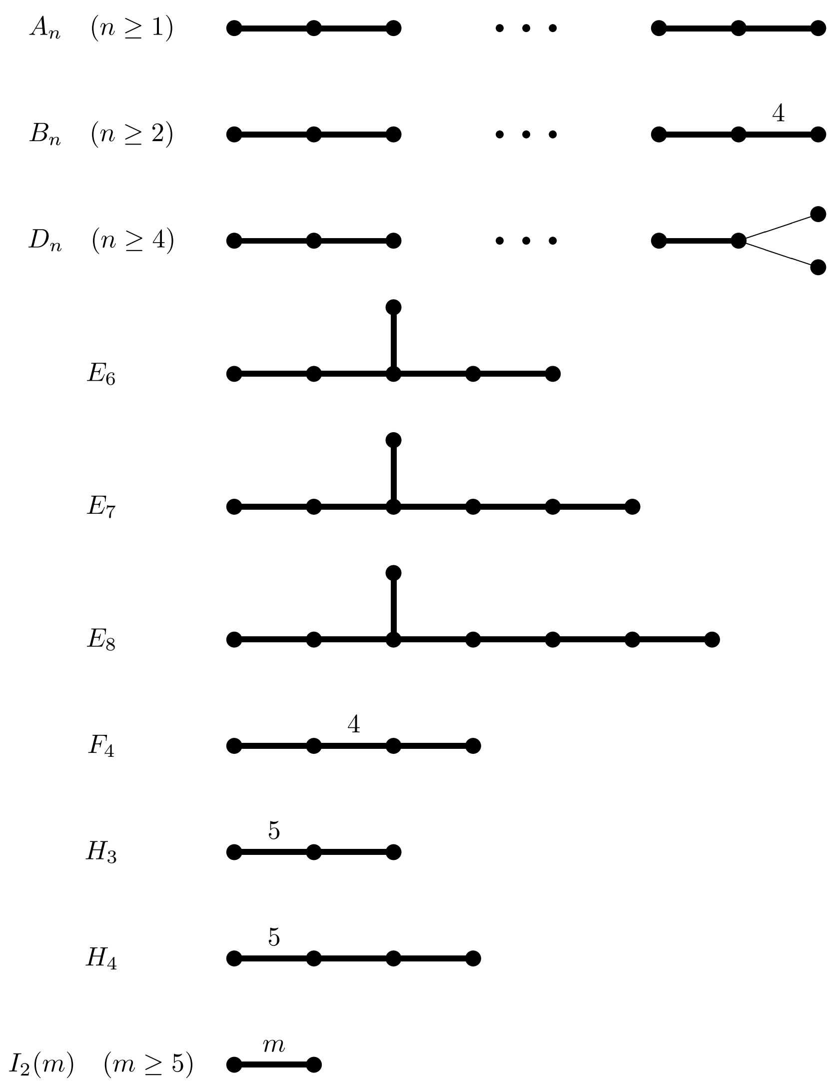


Figure 1: The connected Coxeter graphs of spherical type

The group $\mathcal{A}$ is called the **Artin group** of type Γ (or M), and is sometimes denoted by $\mathcal{A}_\Gamma$. So, as with Coxeter systems, an Artin system is an Artin group with a prescribed set of generators.

There is a natural map $\nu : \mathcal{A}_\Gamma \longrightarrow W_\Gamma$ sending generator $a_i \in \Sigma$ to the corresponding generator $s_i \in \mathcal{S}$. This map is indeed a homomorphism since the equation $\langle s_i s_j \rangle^{m_{ij}} = \langle s_j s_i \rangle^{m_{ij}}$ follows from $s_i^2 = 1$, $s_j^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$. Since ν is clearly surjective it follows that the Coxeter group W_Γ is a quotient of the Artin group $\mathcal{A}_\Gamma$. The kernel of ν is called the **pure Artin group**, generalizing the definition of the pure braid group.

Example 2.3 *The Artin group $\mathcal{A}_{A_n}$ is isomorphic with the braid group $\mathfrak{B}_{n+1}$ on $n+1$ strings. The homomorphism $\mathcal{A}_{A_n} \to W_{A_n}$ corresponds to the map which assigns to each braid the permutation determined by running from one end of the strings to the other. Pure braids are those for which the permutation is the identity.*

The Artin group of a spherical-type Coxeter graph is called an Artin group of **spherical-type**, that is, the corresponding Coxeter group W_Γ is finite. An Artin group $\mathcal{A}_\Gamma$ is called **irreducible** if the Coxeter graph Γ is connected. In particular, the Artin groups corresponding to the graphs in figure 1 are those which are irreducible and of spherical-type. These Artin groups are by far the most well-understood, and are our main interest in the remaining sections.

Van der Lek [Lek83] has shown that for each subgraph $I \subset \Sigma$ the corresponding subgroup and subgraph are an Artin system. That is, parabolic subgroups of Artin groups (those generated by a subset of the generators) are indeed Artin groups. A proof of this fact also appears in [Par97]. Thus inclusions among Coxeter graphs give rise to inclusions for the associated Artin groups. Crisp [Cri99] shows quite a few more inclusions hold among the irreducible spherical-type Artin groups. Table 1 summarizes these inclusions. Notice that every irreducible spherical-type Artin group embeds into an Artin group of type A, D or E.

Cohen and Wales [CW02] use the fact that irreducible finite type Artin groups embed into an Artin group of type A, D or E to show all Artin groups of spherical-type are **linear** (have a faithful finite-dimensional linear representation) by showing Artin groups of type D, and E are linear, thus generalizing the recent result that the braid groups (Artin groups of type A) are linear [Big01], [Kra02].

We close this section by noting that Deligne [Del72] showed that each Artin group of spherical-type appears as the fundamental group of the complement of a complex hyperplane arrangement, which is an Eilenberg-Maclane space. From this point of view we can see that spherical-type Artin groups are torsion free and have finite cohomological dimension.

$\mathcal{A}_\Gamma$ injects into $A_{\Gamma'}$	
Γ	Γ'
A_n	A_m $(m \geq n)$, B_{n+1} $(n \geq 2)$, D_{n+2}, E_6 $(1 \leq n \leq 5)$, E_7 $(1 \leq n \leq 6)$, E_8 $(1 \leq n \leq 7)$, F_4, H_3 $(1 \leq n \leq 2)$, H_4 $(1 \leq n \leq 3)$ $I_2(3)$ $(1 \leq n \leq 2)$
B_n	A_n, A_{2n-1}, A_{2n}, D_{n+1}
E_6	E_7, E_8
E_7	E_8
F_4	E_6, E_7, E_8
H_3	D_6
H_4	E_8
$I_2(m)$	A_{m-1}

Table 1: Inclusions among Artin groups

3 Commutator Subgroups

Our basic tool for finding presentations for the commutator subgroups of Artin groups is the classical Reidemeister-Schreier method. To fix notation, we begin with a brief review of this algorithm. For a more complete discussion, see [MKS76].

3.1 Reidemeister-Schreier algorithm

Let G be an arbitrary group with presentation $\langle a_1, \ldots, a_n : R_\mu(a_\nu), \ldots \rangle$ and H a subgroup of G. A system of words $\mathcal{R}$ in the generators $a_1, \ldots, a_n$ is called a **Schreier system** for G modulo H if (i) every right coset of H in G contains exactly one word of $\mathcal{R}$ (i.e. $\mathcal{R}$ forms a system of right coset representatives), (ii) for each word in $\mathcal{R}$ any initial segment is also in $\mathcal{R}$ (i.e. initial segments of right coset representatives are again right coset representatives). Such a Schreier system always exists, see for example [MKS76]. Suppose now that we have fixed a Schreier system $\mathcal{R}$. For each word W in the generators $a_1, \ldots, a_n$ we let $\overline{W}$ denote the unique representative in $\mathcal{R}$ of the right coset HW. Denote

$$s_{K,a_v} = Ka_v \cdot \overline{Ka_v}^{-1}, \tag{1}$$

for each $K \in \mathcal{R}$ and generator a_v. A theorem of Reidemeister-Schreier (theorem 2.9 in [MKS76]) states that H has presentation

$$\langle s_{K,a_\nu}, \ldots : s_{M,a_\lambda}, \ldots, \tau(KR_\mu K^{-1}), \ldots \rangle \tag{2}$$

where K is an arbitrary Schreier representative, a_v is an arbitrary generator and R_μ is an arbitrary defining relator in the presentation of G, and M is a Schreier representative and a_λ a generator such that

$$Ma_\lambda \approx \overline{Ma_\lambda},$$

where $\approx$ means "freely equal", i.e. equal in the free group generated by $\{a_1, \ldots, a_n\}$. The function τ is a **Reidemeister rewriting function** and is defined according to the rule

$$\tau(a_{i_1}^{\epsilon_1} \cdots a_{i_p}^{\epsilon_p}) = s_{K_{i_1},a_{i_1}}^{\epsilon_1} \cdots s_{K_{i_p},a_{i_p}}^{\epsilon_p} \tag{3}$$

where $K_{i_j} = \overline{a_{i_1}^{\epsilon_1} \cdots a_{i_{j-1}}^{\epsilon_{j-1}}}$, if $\epsilon_j = 1$, and $K_{i_j} = \overline{a_{i_1}^{\epsilon_1} \cdots a_{i_j}^{\epsilon_j}}$, if $\epsilon_j = -1$. It should be noted that computation of $\tau(U)$ can be carried out by replacing a symbol a_v^ϵ of U by the appropriate s-symbol s_{K,a_ν}^ϵ. The main property of a Reidemeister rewriting function is that for an element $U \in H$ given in terms of the generators a_ν the word $\tau(U)$ is the same element of H rewritten in terms of the generators s_{K,a_ν}.

3.2 Characterization of the commutator subgroups

The **commutator subgroup** G' of a group G is the subgroup generated by the elements $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$ for all $g_1, g_2 \in G$. Such elements are called **commutators**. It is an elementary fact in group theory that G' is a normal subgroup in G and the quotient group G/G' is abelian. In fact, for any normal subgroup $N \lhd G$ the quotient group G/N is abelian if and only if $G' < N$. If G is given in terms of a presentation $\langle \mathcal{G} : \mathcal{R} \rangle$ where $\mathcal{G}$ is a set of generators and $\mathcal{R}$ is a set of relations, then a presentation for G/G' is obtained by abelianizing the presentation for G, that is, by adding relations $gh = hg$ for all $g, h \in \mathcal{G}$. This is denoted by $\langle \mathcal{G} : \mathcal{R} \rangle_{\mathrm{Ab}}$.

Let $U \in \mathcal{A}_\Gamma$, and write $U = a_{i_1}^{\epsilon_1} \cdots a_{i_r}^{\epsilon_r}$, where $\epsilon_i = \pm 1$. The (**canonical**) **degree of** U is defined to be

$$\deg(U) := \textstyle\sum_{j=1}^r \epsilon_j.$$

Since each defining relator in the presentation for $\mathcal{A}_\Gamma$ has degree equal to zero the map deg is a well defined homomorphism from $\mathcal{A}_\Gamma$ into $\mathbb{Z}$. Let $\mathcal{N}_\Gamma$ denote the kernel of deg; $\mathcal{N}_\Gamma = \{U \in \mathcal{A}_\Gamma : \deg(U) = 0\}$. It is a well known fact that for the braid group (i.e. $\Gamma = A_n$) $\mathcal{N}_{A_n}$ is precisely the commutator subgroup. In this section we generalize this fact for all Artin groups.

Let Γ_{odd} denote the graph obtained from Γ by removing all the even-labelled edges and the edges labelled ∞. The following theorem tells us exactly when the commutator subgroup $\mathcal{A}'_\Gamma$ is equal to $\mathcal{N}_\Gamma$.

Proposition 3.1 *For an Artin group $\mathcal{A}_\Gamma$, Γ_{odd} is connected if and only if the commutator subgroup $\mathcal{A}'_\Gamma$ is equal to $\mathcal{N}_\Gamma$.*

Proof. For the direction ($\Longrightarrow$) the connectedness of Γ_{odd} implies

$$\mathcal{A}_\Gamma/\mathcal{A}'_\Gamma \simeq \mathbb{Z}.$$

Indeed, start with any generator a_i, for any other generator a_j there is a path from a_i to a_j in Γ_{odd}:

$$a_i = a_{i_i} \longrightarrow a_{i_2} \longrightarrow \cdots \longrightarrow a_{i_m} = a_j.$$

Since $m_{i_k i_{k+1}}$ is odd the relation

$$\langle a_{i_k} a_{i_{k+1}} \rangle^{m_{i_k i_{k+1}}} = \langle a_{i_{k+1}} a_{i_k} \rangle^{m_{i_k i_{k+1}}}$$

becomes $a_{i_k} = a_{i_{k+1}}$ in $\mathcal{A}_\Gamma/\mathcal{A}'_\Gamma$. Hence, $a_i = a_j$ in $\mathcal{A}_\Gamma/\mathcal{A}'_\Gamma$. It follows that,

$$\begin{aligned} \mathcal{A}_\Gamma/\mathcal{A}'_\Gamma &\simeq \langle a_1, \ldots, a_n : a_1 = \cdots = a_n \rangle \\ &\simeq \mathbb{Z}, \end{aligned}$$

where the isomorphism $\phi : \mathcal{A}_\Gamma/\mathcal{A}'_\Gamma \longrightarrow \mathbb{Z}$ is given by

$$U\mathcal{A}'_\Gamma \longmapsto \deg(U).$$

Therefore, $\mathcal{A}'_\Gamma = \ker\phi = \mathcal{N}_\Gamma$.

We leave the proof of the other direction to proposition 3.2, where a more general result is stated. □

For the case when Γ_{odd} is not connected we can get a more general description of $\mathcal{A}'_\Gamma$ as follows. Let Γ_{odd} have m connected components; $\Gamma_{odd} = \Gamma_1 \sqcup \ldots \sqcup \Gamma_m$. Let $\Sigma_i \subset \Sigma$ be the corresponding sets of vertices. For each $1 \leq k \leq m$ define the map

$$\deg_k : \mathcal{A}_\Gamma \longrightarrow \mathbb{Z}$$

as follows: If $U = a_{i_1}^{\epsilon_1} \cdots a_{i_r}^{\epsilon_r} \in \mathcal{A}_\Gamma$ take

$$\deg_k(U) = \sum_{1 \leq j \leq r \text{ } \mathbf{where} \text{ } a_{i_j} \in \Sigma_k} \epsilon_j.$$

It is straight forward to check that for each $1 \leq k \leq m$ the map $\deg_k$ agrees on $\langle ab \rangle^{m_{ab}}$ and $\langle ba \rangle^{m_{ab}}$ for all $a, b \in \Sigma$. Hence, $\deg_k : \mathcal{A}_\Gamma \longrightarrow \mathbb{Z}$ is a homomorphism for each $1 \leq k \leq m$. We combine these m degree maps to get the following homomorphism:

$$\deg_\Gamma : \mathcal{A}_\Gamma \longrightarrow \mathbb{Z}^m$$

by

$$\deg_\Gamma(U) = (\deg_1(U), \ldots, \deg_m(U)).$$

When Γ_{odd} is connected, i.e. $m = 1$, $\deg_\Gamma$ is just the canonical degree. For $U \in \mathcal{A}_\Gamma$ we call $\deg_\Gamma(U)$ the **degree** of U. The following theorem tells us that the kernel of $\deg_\Gamma$ is precisely the commutator subgroup of $\mathcal{A}_\Gamma$.

Proposition 3.2 *Let Γ be a Coxeter graph such that Γ_{odd} has m connected components. Then $\mathcal{A}'_\Gamma = \ker(\deg_\Gamma)$ and $\mathcal{A}_\Gamma/\mathcal{A}'_\Gamma \cong \mathbb{Z}^m$.*

Proof. This follows from

$$\begin{aligned} \mathcal{A}_\Gamma/\mathcal{A}'_\Gamma &\simeq \langle a_1, \ldots, a_n : \langle a_i a_j\rangle^{m_{a_i a_j}} = \langle a_j a_i\rangle^{m_{a_i a_j}}\rangle_{\mathrm{Ab}} \\ &\simeq \langle a_1, \ldots, a_n : a_i = a_j \text{ iff } i \text{ and j lie in the same connected} \\ &\qquad\qquad \text{component of } \Gamma_{odd}\rangle_{\mathrm{Ab}}, \\ &\simeq \mathbb{Z}^m, \end{aligned}$$

with the isomorphism given by

$$U\mathcal{A}'_\Gamma \longmapsto (\deg_1(U), \ldots, \deg_m(U)) = \deg_\Gamma(U),$$

In other words, $\deg_\Gamma$ is precisely the abelianization map on $\mathcal{A}_\Gamma$. □

It follows that $\mathcal{A}_\Gamma$ and $\mathcal{A}'_\Gamma$ fit into a short exact sequence

$$1 \longrightarrow \mathcal{A}'_\Gamma \longrightarrow \mathcal{A}_\Gamma \xrightarrow{\deg_\Gamma} \mathbb{Z}^m \longrightarrow 1,$$

3.3 Computing the presentations

In this section we compute presentations for the commutator subgroups of the irreducible spherical-type Artin groups. We will show that, for the most part, the commutator subgroups are finitely generated and perfect (equal to its commutator subgroup).

Figure 2 shows that each irreducible spherical-type Artin group falls into one of two classes; (i) those in which Γ_{odd} is connected and (ii) those in which Γ_{odd} has two components. Within a given class the arguments are quite similar. Thus, we will only show the complete details of the computations for types A_n and B_n. The rest of the types have similar computations.

3.3.1 Lemmas for simplifying presentations

We will encounter two sets of relations quite often in our computations and it will be necessary to replace them with sets of simpler but equivalent relations. In this section we give two lemmas which allow us to make these replacements.

Let $\{p_k\}_{k\in\mathbb{Z}}$, a, b, and q be letters. In the following keep in mind that the relators $p_{k+1}p_{k+2}^{-1}p_k^{-1}$ split up into the two types of relations $p_{k+2} = p_k^{-1}p_{k+1}$ (for $k \geq 0$), and $p_k = p_{k+1}p_{k+2}^{-1}$ (for $k < 0$). The two lemmas are:

Lemma 3.3 *The set of relations*

$$p_{k+1}p_{k+2}^{-1}p_k^{-1} = 1, \quad p_k a p_{k+2} a^{-1} p_{k+1}^{-1} a^{-1} = 1, \quad b = p_0 a p_0^{-1}, \tag{4}$$

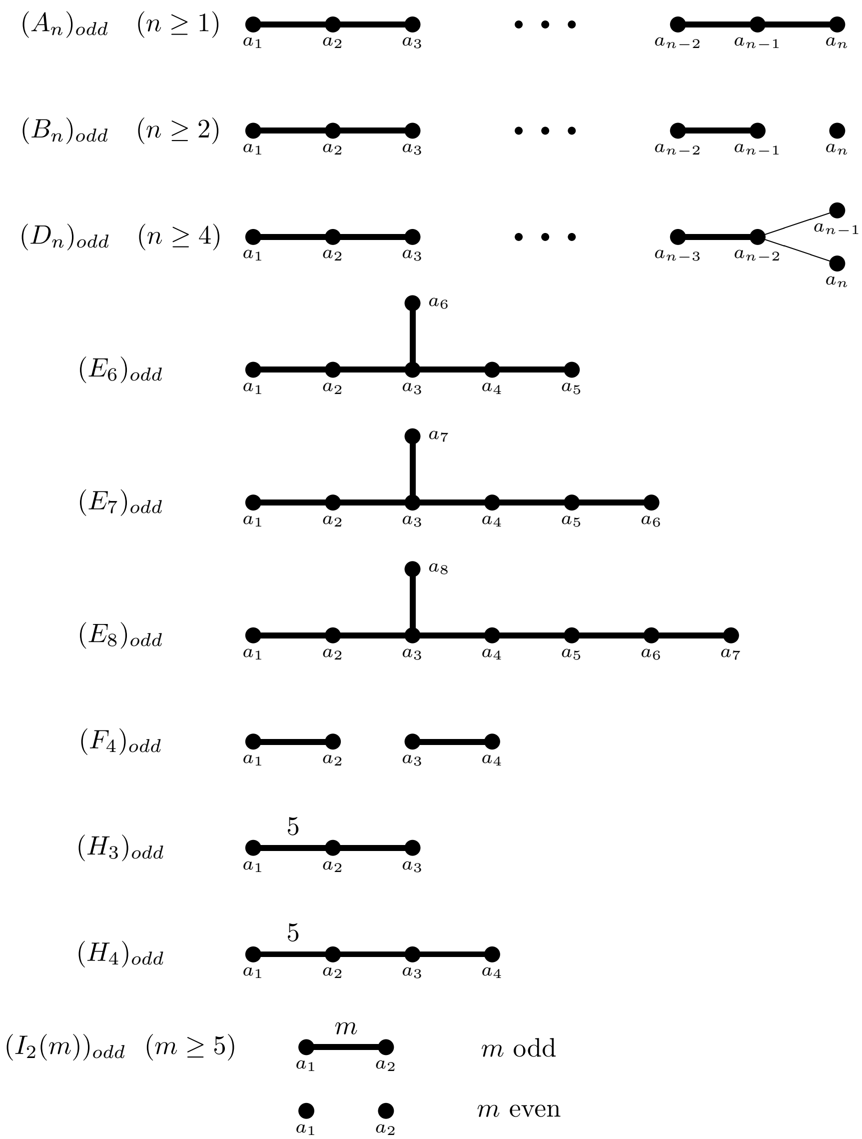


Figure 2: Γ_{odd} for the irreducible spherical-type Coxeter graphs Γ

is equivalent to the set

$$\begin{aligned}
p_{k+1}p_{k+2}^{-1}p_k^{-1} &= 1, && (5)\\
p_0 a p_0^{-1} &= b, && (6)\\
p_0 b p_0^{-1} &= b^2 a^{-1} b && (7)\\
p_1 a p_1^{-1} &= a^{-1} b, && (8)\\
p_1 b p_1^{-1} &= (a^{-1}b)^3 a^{-2} b. && (9)
\end{aligned}$$

Lemma 3.4 *The set of relations:*

$$p_{k+1}p_{k+2}^{-1}p_k^{-1} = 1, \quad p_k q = q p_{k+1},$$

is equivalent to the set

$$p_{k+1}p_{k+2}^{-1}p_k^{-1} = 1, \quad p_0 q = q p_1, \quad p_1 q = q p_0^{-1} p_1.$$

The proof of lemma 3.4 is straightforward. On the other hand, the proof of the lemma 3.3 is somewhat long and tedious.

Proof. [Lemma 3.4] Clearly the second set of relations follows from the first set of relations since $p_2 = p_0^{-1}p_1$. To prove the converse we first prove that $p_k q = q p_{k+1}$ $(k \geq 0)$ follows from the second set of relations by induction on k. It is easy to see then that the same is true for $k < 0$. For $k = 0, 1$ the result clearly holds. Now, for $k = m + 2$;

$$\begin{aligned}
p_{m+2} q p_{m+3}^{-1} q^{-1} &= p_{m+2} q p_{m+2}^{-1} p_{m+1} q^{-1}, \\
&= p_{m+2} (p_{m+1}^{-1} q) p_{m+1} q^{-1} \quad \text{by IH } (k = m+1), \\
&= p_{m+2} p_{m+1}^{-1} (q p_{m+1}) q^{-1}, \\
&= p_{m+2} p_{m+1}^{-1} (p_m q) q^{-1} \quad \text{by IH } (k = m), \\
&= p_{m+2} p_{m+1}^{-1} p_m, \\
&= 1.
\end{aligned}$$

□

Proof. [Lemma 3.3] First we show the second set of relations follows from the first set. Taking $k = 0$ in the second relation in (4) we get the relation

$$p_0 a p_2 a^{-1} p_1^{-1} a^{-1} = 1,$$

and, using the relations $p_2 = p_0^{-1}p_1$ and $b = p_0 a p_0^{-1}$, (8) easily follows. Taking $k = 1$ in the second relation in (4) we get the relation

$$p_1 a p_3 a^{-1} p_2^{-1} a^{-1} = 1.$$

Using the relations $p_3 = p_1^{-1}p_2$ and $p_2 = p_0^{-1}p_1$ this becomes

$$p_1 a p_1^{-1} p_0^{-1} p_1 a^{-1} p_1^{-1} p_0 a^{-1} = 1.$$

But $p_1ap_1^{-1} = a^{-1}b$ (by (8)) so this reduces to

$$a^{-1}bp_0^{-1}b^{-1}ap_0a^{-1} = 1.$$

Isolating bp_0^{-1} on one side of the equation gives

$$bp_0^{-1} = a^2p_0^{-1}a^{-1}b.$$

Multiplying both sides on the left by p_0 and using the relation $p_0ap_0^{-1} = b$ it easily follows $p_0bp_0^{-1} = b^2a^{-1}b$, which is (7). Finally, taking $k = 2$ in the second relation in (4) we get the relation

$$p_2ap_4a^{-1}p_3^{-1}a^{-1} = 1.$$

Using the relation $p_4 = p_2^{-1}p_3$ this becomes

$$p_2ap_2^{-1}p_3a^{-1}p_3^{-1}a^{-1} = 1. \tag{10}$$

Note that

$$\begin{aligned} p_2ap_2^{-1} &= p_0^{-1}p_1ap_1^{-1}p_0 \quad \text{by } p_2 = p_0^{-1}p_1 \\ &= p_0^{-1}a^{-1}bp_0 \quad \text{by (8)} \\ &= a^{-2}ba^{-1}a \quad \text{using (4) and (7)} \\ &= a^{-2}b \end{aligned}$$

and

$$\begin{aligned} p_3ap_3^{-1} &= p_1^{-1}p_2ap_2^{-1}p_1 \qquad \text{by } \mathrm{p}_3 = \mathrm{p}_1^{-1}\mathrm{p}_2 \\ &= p_1^{-1}a^{-2}bp_1, \end{aligned}$$

where the second equality follows from the previous statement. Thus, (10) becomes

$$a^{-2}bp_1^{-1}b^{-1}a^2p_1a^{-1} = 1$$

Isolating the factor bp_1^{-1} on one side of the equation, multiplying both sides by p_1, and using the relation (8) we easily get the relation (9). Therefore we have that the second set of relations (5)-(9) follows from the first set of relations (4).

In order to show the relations in (4) follow from the relations in (5)-(9) it suffices to just show that the second relation in (4) follows from the relations in (5)-(9). To do this we need the following fact: The relations

$$p_kap_k^{-1} = a^kb, \tag{11}$$
$$p_kbp_k^{-1} = (a^{-k}b)^{k+2}a^{-(k+1)}b, \tag{12}$$
$$p_k^{-1}ap_k = ab^{-1}a^{k+2}, \tag{13}$$
$$p_k^{-1}bp_k = (ab^{-1}a^{k+2})^ka, \tag{14}$$

follow from the relations in (5)-(9). The proof of this fact is left to lemma 3.5 below. From the relations (11)-(14) we obtain

$$p_{k+1}ap_{k+1}^{-1} = a^{-(k+1)}b = a^{-1} \cdot a^{-k}b = a^{-1}p_kap_k^{-1}, \tag{15}$$

and

$$p_{k+1}^{-1}ap_{k+1} = ab^{-1}a^{k+3} = ab^{-1}a^{k+2}a = p_k^{-1}ap_ka. \tag{16}$$

Now we are in a position to show that that the second relation in (4) follows from the relations in (5)-(9). For $k \geq 0$

$$\begin{aligned} p_kap_{k+2}a^{-1}p_{k+1}^{-1}a^{-1} &= p_kap_k^{-1}\underbrace{p_{k+1}a^{-1}p_{k+1}^{-1}}a^{-1} \qquad \text{by (5)} \\ &= p_kap_k^{-1}(a^{-1}p_kap_k^{-1})^{-1}a^{-1} \qquad \text{by (15)} \\ &= 1. \end{aligned}$$

and for $k < 0$

$$\begin{aligned} p_kap_{k+2}a^{-1}p_{k+1}^{-1}a^{-1} &= p_{k+1}\underbrace{p_{k+2}^{-1}ap_{k+2}}a^{-1}p_{k+1}^{-1}a^{-1} \qquad \text{by (5)} \\ &= p_{k+1}(p_{k+1}^{-1}ap_{k+1}a)a^{-1}p_{k+1}^{-1}a^{-1} \qquad \text{by (16)} \\ &= 1. \end{aligned}$$

Therefore, the relations

$$p_kap_{k+2}a^{-1}p_{k+1}^{-1}a^{-1} = 1, \qquad k \in \mathbb{Z}$$

follow from the relations in (5)-(9). □

To complete the proof of lemma 3.3 we need to prove the following.

Lemma 3.5 *The relations*

$$\begin{aligned} p_kap_k^{-1} &= a^kb \\ p_kbp_k^{-1} &= (a^{-k}b)^{k+2}a^{-(k+1)}b \\ p_k^{-1}ap_k &= ab^{-1}a^{k+2} \\ p_k^{-1}bp_k &= (ab^{-1}a^{k+2})^ka \end{aligned}$$

follow from the relations in (5)-(9).

Proof. We will use induction to prove the result for nonnegative indices k, the result for negative indices k is similar. Clearly this holds for $k = 0, 1$. For

$k = m+2$ we have

$$\begin{aligned}
p_{m+2} a p_{m+2}^{-1} &= p_m^{-1} p_{m+1} a p_{m+1}^{-1} p_m \quad \text{by (5)},\\
&= p_m^{-1} a^{-(m+1)} b p_m \quad \text{by induction hypothesis (IH)},\\
&= (p_m^{-1} a^{-(m+1)} p_m)(p_m^{-1} b p_m),\\
&= (p_m^{-1} a p_m)^{-(m+1)}(p_m^{-1} b p_m),\\
&= (ab^{-1}a^{m+2})^{-(m+1)}(ab^{-1}a^{m+2})^m a \quad \text{by IH},\\
&= (ab^{-1}a^{m+2})^{-1} a,\\
&= a^{-(m+2)} b,
\end{aligned}$$

$$\begin{aligned}
p_{m+2} b p_{m+2}^{-1} &= p_m^{-1} p_{m+1} b p_{m+1}^{-1} p_m \quad \text{by (5)},\\
&= p_m^{-1}(a^{-(m+1)}b)^{m+3} a^{-(m+2)} b p_m \quad \text{by IH},\\
&= ((p_m^{-1} a p_m)^{-(m+1)}(p_m^{-1} b p_m))^{m+3}(p_m^{-1} a p_m)^{-(m+2)} p_m^{-1} b p_m,\\
&= ((ab^{-1}a^{m+2})^{-(m+1)}(ab^{-1}a^{m+2})^m a)^{(m+3)}\\
&\qquad \cdot (ab^{-1}a^{m+2})^{-(m+2)}(ab^{-1}a^{m+2})^m a \quad \text{by IH},\\
&= (a^{-(m+2)}b)^{m+3}(ab^{-1}a^{m+2})^{-2} a,\\
&= (a^{-(m+2)}b)^{m+4} a^{-(m+3)} b,
\end{aligned}$$

Similarly for the other two equations. Thus, the result follows by induction. □

3.3.2 Type A

The first presentation for the commutator subgroup $\mathfrak{B}'_{n+1} = \mathcal{A}'_{A_n}$ of the braid group $\mathfrak{B}_{n+1} = \mathcal{A}_{A_n}$ appeared in [GL69] but the details of the computation were minimal. Here we fill in the details of Gorin and Lin's computation.

The presentation of $\mathcal{A}_{A_n}$ is

$$\begin{aligned}
\mathcal{A}_{A_n} = \langle a_1, ..., a_n : \quad & a_i a_j = a_j a_i \quad \text{for } |i-j| \geq 2,\\
& a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad \text{for } 1 \leq i \leq n-1 \ \rangle.
\end{aligned}$$

Since $(A_n)_{odd}$ is connected then by proposition 3.2 $\mathcal{A}'_{A_n} = \ker(\deg)$. Elements $U, V \in \mathcal{A}_{A_n}$ lie in the same right coset of $\mathcal{A}'_{A_n}$ precisely when they have the same degree:

$$\begin{aligned}
\mathcal{A}'_{A_n} U = \mathcal{A}'_{A_n} V \quad &\Longleftrightarrow \quad UV^{-1} \in \mathcal{A}'_{A_n}\\
&\Longleftrightarrow \quad \deg(U) = \deg(V),
\end{aligned}$$

thus a Schreier system of right coset representatives for $\mathcal{A}_{A_n}$ modulo $\mathcal{A}'_{A_n}$ is

$$\mathcal{R} = \{a_1^k : k \in \mathbb{Z}\}$$

By the Reidemeister-Schreier method, in particular equation (2), $\mathcal{A}'_{A_n}$ has generators $s_{a_1^k, a_j} := a_1^k a_j (\overline{a_1^k a_j})^{-1}$ with presentation

$$\langle s_{a_1^k, a_j}, \ldots : s_{a_1^m, a_\lambda}, \ldots, \tau(a_1^\ell R_i a_1^{-\ell}), \ldots, \tau(a_1^\ell T_{i,j} a_1^{-\ell}), \ldots \rangle, \tag{17}$$

where $j \in \{1, \ldots, n\}$, $k, \ell \in \mathbb{Z}$, and $m \in \mathbb{Z}$, $\lambda \in \{1, \ldots, n\}$ such that $a_1^m a_\lambda \approx \overline{a_1^m a_\lambda}$ ("freely equal"), and $T_{i,j}$, R_i represent the relators $a_i a_j a_i^{-1} a_j^{-1}$, $|i-j| \geq 2$, and $a_i a_{i+1} a_i a_{i+1}^{-1} a_i^{-1} a_{i+1}^{-1}$, respectively. Our goal is to clean up this presentation.

The first thing to notice is that

$$a_1^m a_\lambda \approx \overline{a_1^m a_\lambda} = a_1^{m+1} \Longleftrightarrow \lambda = 1$$

Thus, the first type of relation in (17) is precisely $s_{a_1^m, a_1} = 1$, for all $m \in \mathbb{Z}$.

Next, we use the definition of the Reidemeister rewriting function (3) to express the second and third types of relations in (17) in terms of the generators $s_{a_1^k, a_j}$:

$$\begin{aligned} \tau(a_1^k T_{i,j} a_1^{-k}) &= s_{a_1^k, a_i} s_{a_1^{k+1}, a_j} s^{-1}_{a_1^{k+1}, a_i} s^{-1}_{a_1^k, a_j} && (18) \\ \tau(a_1^k R_i a_1^{-k}) &= s_{a_1^k, a_i} s_{a_1^{k+1}, a_{i+1}} s_{a_1^{k+2}, a_i} s^{-1}_{a_1^{k+2}, a_{i+1}} s^{-1}_{a_1^{k+1}, a_i} s^{-1}_{a_1^k, a_{i+1}} && (19) \end{aligned}$$

Taking $i = 1$, $j \geq 3$ in (18) we get

$$s_{a_1^{k+1}, a_j} = s_{a_1^k, a_j}$$

Thus, by induction on k,

$$s_{a_1^k, a_j} = s_{1, a_j} \qquad (20)$$

for $j \geq 3$ and for all $k \in \mathbb{Z}$.

Therefore, $\mathcal{A}'_{A_n}$ is generated by $s_{a_1^k, a_2} = a_1^k a_2 a_1^{-(k+1)}$ and $s_{1, a_\ell} = a_\ell a_1^{-1}$, where $k \in \mathbb{Z}$, $3 \leq \ell \leq n$. To simplify notation let us rename the generators; let $p_k := a_1^k a_2 a_1^{-(k+1)}$ and $q_\ell := a_\ell a_1^{-1}$, for $k \in \mathbb{Z}$, $3 \leq \ell \leq n$. We now investigate the relations in (18) and (19).

The relations in (19) break up into the following three types (using 20):

$$\begin{aligned} & p_{k+1} p_{k+2}^{-1} p_k^{-1} && \text{(taking } i = 1\text{)} && (21) \\ & p_k q_3 p_{k+2} q_3^{-1} p_{k+1}^{-1} q_3^{-1} && \text{(taking } i = 2\text{)} && (22) \\ & q_i q_{i+1} q_i q_{i+1}^{-1} q_i^{-1} q_{i+1}^{-1} && \text{for } 3 \leq i \leq n-1. && (23) \end{aligned}$$

The relations in (18) break up into the following two types:

$$\begin{aligned} & p_k q_j p_{k+1}^{-1} q_j^{-1} && \text{for } 4 \leq j \leq n, \text{ (taking } i = 2\text{)} && (24) \\ & q_i q_j q_i^{-1} q_j^{-1} && \text{for } 3 \leq i < j \leq n,\ |i-j| \geq 2. && (25) \end{aligned}$$

We now have a presentation for $\mathcal{A}'_{A_n}$ consisting of the generators p_k, q_ℓ, where $k \in \mathbb{Z}$, $3 \leq \ell \leq n-1$, and defining relations (21) -(25). However, notice that relation (21) splits up into the two relations

$$\begin{aligned} p_{k+2} &= p_k^{-1} p_{k+1} && \text{for } k \geq 0, && (26) \\ p_k &= p_{k+1} p_{k+2}^{-1} && \text{for } k < 0. && (27) \end{aligned}$$

Thus, for $k \neq 0, 1$, p_k can be expressed in terms of p_0 and p_1. It follows that $\mathcal{A}'_{A_n}$ is finitely generated. In order to show $\mathcal{A}'_{A_n}$ is finitely presented we need to be able to replace the infinitly many relations in (22) and (24) with finitely many relations. This can be done using lemmas 3.3 and 3.4, but this requires us to add a new letter b to the generating set with a new relation $b = p_0 q_3 p_0^{-1}$. Thus $\mathcal{A}'_{A_n}$ is generated by p_0, p_1, q_ℓ, b, where $3 \leq \ell \leq n-1$, with defining relations:

$$p_0 q_3 p_0^{-1} = b, \quad p_0 b p_0^{-1} = b^2 q_3^{-1} b, \quad p_1 q_3 p_1^{-1} = q_3^{-1} b, \quad p_1 b p_1^{-1} = (q_3^{-1} b)^3 q_3^{-2} b,$$
$$q_i q_{i+1} q_i q_{i+1}^{-1} q_i^{-1} q_{i+1}^{-1} \quad (3 \leq i \leq n-1),$$
$$p_0 q_j = q_j p_1 \quad (4 \leq j \leq n), \quad p_1 q_j = q_j p_0^{-1} p_1 \quad (4 \leq j \leq n).$$
$$q_i q_j q_i^{-1} q_j^{-1} \qquad (3 \leq i < j \leq n,\ |i-j| \geq 2).$$

Noticing that for $n = 2$ the generators q_k $(3 \leq k \leq n)$, and b do not exist, and for $n = 3$ the generators q_k $(4 \leq k \leq n)$ do not exist, we have proved the following theorem.

Theorem 3.6 *For every $n \geq 2$ the commutator subgroup $\mathcal{A}'_{A_n}$ of the Artin group $\mathcal{A}_{A_n}$ is a finitely presented group. $\mathcal{A}'_{A_2}$ is a free group with two free generators*

$$p_0 = a_2 a_1^{-1}, \quad p_1 = a_1 a_2 a_1^{-2}.$$

$\mathcal{A}'_{A_3}$ is the group generated by

$$p_0 = a_2 a_1^{-1}, \quad p_1 = a_1 a_2 a_1^{-2}, \quad q = a_3 a_1^{-1}, \quad b = a_2 a_1^{-1} a_3 a_2^{-1},$$

with defining relations

$$b = p_0 q p_0^{-1}, \quad p_0 b p_0^{-1} = b^2 q^{-1} b,$$
$$p_1 q p_1^{-1} = q^{-1} b, \quad p_1 b p_1^{-1} = (q^{-1} b)^3 q^{-2} b.$$

For $n \geq 4$ the group $\mathcal{A}'_{A_n}$ is generated by

$$p_0 = a_2 a_1^{-1}, \quad p_1 = a_1 a_2 a_1^{-2}, \quad q_3 = a_3 a_1^{-1},$$
$$b = a_2 a_1^{-1} a_3 a_2^{-1}, \quad q_\ell = a_\ell a_1^{-1} \quad (4 \leq \ell \leq n-1),$$

with defining relations

$$b = p_0 q_3 p_0^{-1}, \quad p_0 b p_0^{-1} = b^2 q_3^{-1} b,$$
$$p_1 q_3 p_1^{-1} = q_3^{-1} b, \quad p_1 b p_1^{-1} = (q_3^{-1} b)^3 q_3^{-2} b,$$
$$p_0 q_i = q_i p_1 \quad (4 \leq i \leq n), \quad p_1 q_i = q_i p_0^{-1} p_1 \quad (4 \leq i \leq n)$$
$$q_3 q_i = q_i q_3 \quad (5 \leq i \leq n), \quad q_3 q_4 q_3 = q_4 q_3 q_4,$$
$$q_i q_j = q_j q_i \quad (4 \leq i < j-1 \leq n-1), \quad q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \quad (4 \leq i \leq n-1).$$

$\square$

Corollary 3.7 *For $n \geq 4$ the commutator subgroup $\mathcal{A}'_{A_n}$ of the Artin group of type A_n is finitely generated and perfect (i.e. $\mathcal{A}''_{A_n} = \mathcal{A}'_{A_n}$).*

Proof. Abelianizing the presentation of $\mathcal{A}'_{A_n}$ in the theorem results in a presentation of the trivial group. Hence $\mathcal{A}''_{A_n} = \mathcal{A}'_{A_n}$. □

Now we study in greater detail the group $\mathcal{A}'_{A_3}$, the results of which will be used in section 4.2.1. From the presentation of $\mathcal{A}'_{A_3}$ given in theorem 3.6 one can easily deduce the relations:

$$p_0^{-1}qp_0 = qb^{-1}q^2, \quad p_0^{-1}bp_0 = q,$$
$$p_1^{-1}qp_1 = qb^{-1}q^3, \quad p_1^{-1}bp_1 = qb^{-1}q^4.$$

Let T be the subgroup of $\mathcal{A}'_{A_3}$ generated by q and b. The above relations and the defining relations in the presentation for $\mathcal{A}'_{A_3}$ tell us that T is a normal subgroup of $\mathcal{A}'_{A_3}$. To obtain a representation of the factor group $\mathcal{A}'_{A_3}/T$ it is sufficient to add to the defining relations in the presentation for $\mathcal{A}'_{A_3}$ the relations $q = 1$ and $b = 1$. It is easy to see this results in the presentation of the free group generated by p_0 and p_1. Thus, $\mathcal{A}'_{A_3}/T$ is a free group of rank 2, F_2. We have the exact sequence

$$1 \longrightarrow T \longrightarrow \mathcal{A}'_{A_3} \longrightarrow \mathcal{A}'_{A_3}/T \longrightarrow 1.$$

Since $\mathcal{A}'_{A_3}/T$ is free then the exact sequence is actually split so

$$\mathcal{A}'_{A_3} \simeq T \rtimes \mathcal{A}'_{A_3}/T \simeq T \rtimes F_2,$$

where the action of F_2 on T is given by the defining relations in the presentation of $\mathcal{A}'_{A_3}$ and the relations above. In [GL69] it is shown (theorem 2.6) the group T is also free of rank 2, so we have the following theorem.

Theorem 3.8 *The commutator subgroup $\mathcal{A}'_{A_3}$ of the Artin group of type A_3 is the semidirect product of two free groups each of rank 2;*

$$\mathcal{A}'_{A_3} \simeq F_2 \rtimes F_2.$$

□

3.3.3 Type B

The presentation of $\mathcal{A}_{B_n}$ is

$$\mathcal{A}_{B_n} = \langle a_1, ..., a_n : \quad \begin{array}{l} a_i a_j = a_j a_i \quad \text{for } |i-j| \geq 2, \\ a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad \text{for } 1 \leq i \leq n-2 \\ a_{n-1} a_n a_{n-1} a_n = a_n a_{n-1} a_n a_{n-1} \rangle. \end{array}$$

Let $T_{i,j}$, R_i $(1 \leq i \leq n-2)$, and R_{n-1} denote the associated relators $a_i a_j a_i^{-1} a_j^{-1}$, $a_i a_{i+1} a_i a_{i+1}^{-1} a_i^{-1} a_{i+1}^{-1}$, and $a_{n-1} a_n a_{n-1} a_n a_{n-1}^{-1} a_n^{-1} a_{n-1}^{-1} a_n^{-1}$, respectively.

As seen in figure 2 the graph $(B_n)_{odd}$ has two components: Γ_1 and Γ_2, where Γ_2 denotes the component containing the single vertiex a_n. Let $\deg_1$ and $\deg_2$ denote the associated degree maps, respectively, so from proposition 3.2

$$\mathcal{A}'_{B_n} = \{U \in \mathcal{A}_{B_n} : \deg_1(U) = 0 \text{ and } \deg_2(U) = 0\}.$$

For elements $U, V \in \mathcal{A}_{A_n}$,

$$\begin{aligned} \mathcal{A}'_{B_n} U = \mathcal{A}'_{B_n} V \quad &\Leftrightarrow \quad UV^{-1} \in \mathcal{A}'_{B_n} \\ &\Leftrightarrow \quad \deg_1(U) = \deg_1(V), \text{ and} \\ & \quad \deg_2(U) = \deg_2(V), \end{aligned}$$

thus a Schreier system of right coset representatives for $\mathcal{A}_{B_n}$ modulo $\mathcal{A}'_{B_n}$ is

$$\mathcal{R} = \{a_1^k a_n^\ell : k, \ell \in \mathbb{Z}\}$$

By the Reidemeister-Schreier method, in particular equation (2), $\mathcal{A}'_{B_n}$ is generated by

$$\begin{aligned} s_{a_1^k a_n^k, a_j} \quad &:= \quad a_1^k a_n^\ell a_j (\overline{a_1^k a_n^\ell a_j})^{-1} \\ &= \quad \begin{cases} a_1^k a_n^\ell a_j a_n^{-\ell} a_1^{-(k+1)} & \text{if } j \neq n \\ 1 & \text{if } j = n. \end{cases} \end{aligned}$$

with presentation

$$\begin{aligned} \mathcal{A}'_{B_n} = \langle s_{a_1^k a_n^\ell, a_j}, \dots : \; & s_{a_1^p a_n^q, a_\lambda}, \dots, \\ & \tau(a_1^k a_n^\ell T_{i,j} (a_1^k a_n^\ell)^{-1}), \dots, \quad (1 \leq i < j \leq n, |i-j| \geq 2), \\ & \tau(a_1^k a_n^\ell R_i (a_1^k a_n^\ell)^{-1}), \dots, \quad (1 \leq i \leq n-2), \\ & \tau(a_1^k a_n^\ell R_{n-1} (a_1^k a_n^\ell)^{-1}), \dots \rangle, \end{aligned} \tag{28}$$

where $p, q \in \mathbb{Z}$, $\lambda \in \{1, \dots, n-1\}$ such that $a_1^p a_n^q a_\lambda \approx \overline{a_1^p a_n^q a_\lambda}$ ("freely equal"). Again, our goal is to clean up this presentation.

The cases $n = 2, 3$, and 4 are straightforward after one sees the computation for the general case $n \geq 5$, so we will not include the computations for these cases. The results are included in theorem 3.9. From now on it will be assumed that $n \geq 5$.

Since

$$a_1^p a_n^q a_\lambda \approx \overline{a_1^p a_n^q a_\lambda} = \begin{cases} a_1^{p+1} a_n^q & \lambda \neq n \\ a_1^p a_n^{q+1} & \lambda = n \end{cases} \iff \lambda = n \text{ or; } \lambda = 1 \text{ and } q = 0,$$

the first type of relations in (28) are precisely

$$s_{a_1^k a_n^\ell, a_n} = 1, \quad \text{and} \quad s_{a_1^k, a_1} = 1. \tag{29}$$

The second type of relations in (28), after rewriting using equation (3), are

$$s_{a_1^k a_n^\ell, a_i} s_{\overline{a_1^k a_n^\ell a_i}, a_j} s^{-1}_{\overline{a_1^k a_n^\ell a_i a_j a_i^{-1}}, a_i} s^{-1}_{\overline{a_1^k a_n^\ell a_i a_j a_i^{-1} a_j^{-1}}, a_j}. \tag{30}$$

where $1 \leq i < j \leq n$, $|i-j| \geq 2$. Taking $i = 1$ and $3 \leq j \leq n-1$ gives: for $\ell = 0$ (using (29));

$$s_{a_1^{k+1}, a_j} = s_{a_1^k, a_j}, \tag{31}$$

so by induction on k,

$$s_{a_1^k, a_j} = s_{1, a_j} \quad \text{for } 3 \leq j \leq n-1, \tag{32}$$

and for $\ell \neq 0$;

$$s_{a_1^k a_n^\ell, a_1} s_{a_1^{k+1} a_n^\ell, a_j} s^{-1}_{a_1^{k+1} a_n^\ell, a_1} s^{-1}_{a_1^k a_n^\ell, a_j}. \tag{33}$$

We will come back to relation (33) in a bit.

Taking $i = 1$ and $j = n$ in (30) (and using (29)) gives

$$s_{a_1^k a_n^\ell, a_1} s^{-1}_{a_1^k a_n^{\ell+1}, a_1}. \tag{34}$$

So, by induction on ℓ (and (29)) we get

$$s_{a_1^k a_n^\ell, a_1} = 1 \quad \text{for } k, \ell \in \mathbb{Z}. \tag{35}$$

Taking $2 \leq i \leq n-2$, $i+2 \leq j \leq n$ in (30) gives

$$\begin{cases} s_{a_1^k a_n^\ell, a_i} s_{a_1^{k+1} a_n^\ell, a_j} s^{-1}_{a_1^{k+1} a_n^\ell, a_i} s^{-1}_{a_1^k a_n^\ell, a_j} & \text{for } j \leq n-1, \\ s_{a_1^k a_n^\ell, a_i} s^{-1}_{a_1^k a_n^{\ell+1}, a_i} & \text{for } j = n. \end{cases} \tag{36}$$

In the case $j = n$ induction on ℓ gives

$$s_{a_1^k a_n^\ell, a_i} = s_{a_1^k, a_i} \quad (2 \leq i \leq n-2). \tag{37}$$

So from (32) it follows

$$s_{a_1^k a_n^\ell, a_i} = \begin{cases} s_{1, a_i} & 3 \leq i \leq n-2 \\ s_{a_1^k, a_2} & i = 2. \end{cases} \tag{38}$$

We come back to the case $j \leq n-1$ later.

Returning now to (33), we can use (35) to get

$$s_{a_1^{k+1} a_n^\ell, a_j} = s_{a_1^k a_n^\ell . a_j} \quad (3 \leq j \leq n-1).$$

Thus, by induction on k

$$s_{a_1^k a_n^\ell, a_j} = s_{a_n^\ell, a_j} \quad (3 \leq j \leq n-1). \tag{39}$$

For $3 \le j \le n-2$ we already know this (equation (38)), so the only new information we get from (33) is

$$s_{a_1^k a_n^\ell, a_{n-1}} = s_{a_n^\ell, a_{n-1}} \quad (k \in \mathbb{Z}). \tag{40}$$

Collecting all the information we have obtained from $\tau(a_1^k a_n^\ell T_{i,j}(a_1^k a_n^\ell)^{-1})$, $1 \le i < j \le n, |i-j| \ge 2$, we get:

$$\begin{aligned} s_{a_1^k a_n^\ell, a_1} &= 1 \quad (k, \ell \in \mathbb{Z}),\\ s_{a_1^k a_n^\ell, a_i} &= \begin{cases} s_{1,a_i} & 3 \le i \le n-2,\\ s_{a_1^k, a_2} & i = 2, \end{cases}\\ s_{a_1^k a_n^\ell, a_{n-1}} &= s_{a_n^\ell, a_{n-1}}, \end{aligned} \tag{41}$$

and (from (36)), for $2 \le i \le n-3$ and $i+2 \le j \le n-1$,

$$s_{a_1^k a_n^\ell, a_i} s_{a_1^{k+1} a_n^\ell, a_j} s^{-1}_{a_1^{k+1} a_n^\ell, a_i} s^{-1}_{a_1^k a_n^\ell, a_j}. \tag{42}$$

This relation breaks up into the following cases (using (41))

$$\begin{cases} s_{a_1^k, a_2} s_{1,a_j} s^{-1}_{a_1^{k+1}, a_2} s^{-1}_{1,a_j} & \text{for } i=2,\ 4 \le j \le n-2,\\ s_{a_1^k, a_2} s_{a_n^\ell, a_{n-1}} s^{-1}_{a_1^{k+1}, a_2} s^{-1}_{a_n^\ell, a_{n-1}} & \text{for } i=2,\ j=n-1,\\ s_{1,a_i} s_{1,a_j} s^{-1}_{1,a_i} s^{-1}_{1,a_j} & \text{for } 3 \le i \le n-3,\ i+2 \le j \le n-2,\\ s_{1,a_i} s_{a_n^\ell, a_{n-1}} s^{-1}_{1,a_i} s^{-1}_{a_n^\ell, a_{n-1}} & \text{for } 3 \le i \le n-3,\ j=n-1, \end{cases} \tag{43}$$

The third type of relations in (28); $\tau(a_1^k a_n^\ell R_i (a_1^k a_n^\ell)^{-1})$, after rewriting using equation (3), are

$$s_{a_1^k a_n^\ell, a_i} s_{a_1^{k+1} a_n^\ell, a_{i+1}} s_{a_1^{k+2} a_n^\ell, a_i} s^{-1}_{a_1^{k+2} a_n^\ell, a_{i+1}} s^{-1}_{a_1^{k+1} a_n^\ell, a_i} s^{-1}_{a_1^k a_n^\ell, a_{i+1}}, \tag{44}$$

which break down as follows (using (41)):

$$\begin{cases} s_{a_1^{k+1}, a_2} s^{-1}_{a_1^{k+2}, a_2} s^{-1}_{a_1^k, a_2} & (i=1),\\ s_{a_1^k, a_2} s_{1,a_3} s_{a_1^{k+2}, a_2} s^{-1}_{1,a_3} s^{-1}_{a_1^{k+1}, a_2} s^{-1}_{1,a_3} & (i=2),\\ s_{1,a_i} s_{1,a_{i+1}} s_{1,a_i} s^{-1}_{1,a_{i+1}} s^{-1}_{1,a_i} s^{-1}_{1,a_{i+1}}, & \text{for } 3 \le i \le n-3,\\ s_{1,a_{n-2}} s_{a_n^\ell, a_{n-1}} s_{1,a_{n-2}} s^{-1}_{a_n^\ell, a_{n-1}} s^{-1}_{1,a_{n-2}} s^{-1}_{a_n^\ell, a_{n-1}}, & (i=n-2), \end{cases} \tag{45}$$

The fourth type of relations in (28); $\tau(a_1^k a_n^\ell R_{n-1} (a_1^k a_n^\ell)^{-1})$, after rewriting using equation (3), is

$$s_{a_n^\ell, a_{n-1}} s_{a_n^{\ell+1}, a_{n-1}} s^{-1}_{a_n^{\ell+2}, a_{n-1}} s^{-1}_{a_n^{\ell+1}, a_{n-1}}, \tag{46}$$

where we have made extensive use of the relations (41).

From (41) it follows that $\mathcal{A}'_{B_n}$ is generated by $s_{a_1^k, a_2}$, s_{1,a_i}, and $s_{a_n^\ell, a_{n-1}}$ for $k, \ell \in \mathbb{Z}$ and $3 \le i \le n-2$. For simplicity of notation let these generators be

denoted by p_k, q_i, and r_ℓ, respectively. Thus, we have shown that the following is a set of defining relations for $\mathcal{A}'_{B_n}$:

$$\begin{aligned}
p_kq_j &= q_jp_{k+1} && (4 \le j \le n-2,\ k \in \mathbb{Z}),\\
p_kr_\ell &= r_\ell p_{k+1} && (k, \ell \in \mathbb{Z}),\\
q_iq_j &= q_jq_i && (3 \le i < j \le n-2,\ |i = j| \ge 2),\\
q_ir_\ell &= r_\ell q_i && (3 \le i \le n-3),\\
& p_{k+1}p_{k+2}^{-1}p_k^{-1} && (k \in \mathbb{Z}),\\
& p_kq_3p_{k+2}q_3^{-1}p_{k+1}^{-1}q_3^{-1} && (k \in Z),\\
q_iq_{i+1}q_i &= q_{i+1}q_iq_{i+1} && (3 \le i \le n-3),\\
q_{n-2}r_\ell q_{n-2} &= r_\ell q_{n-2}r_\ell && (\ell \in \mathbb{Z}),\\
& r_\ell r_{\ell+1}r_{\ell+2}^{-1}r_{\ell+1}^{-1} && (\ell \in \mathbb{Z}),
\end{aligned} \tag{47}$$

The first four relations are from (43), the next four are from (45), and the last one is from (46).

The fifth relation tells us that for $k \ne 0, 1$, p_k can be expressed in terms of p_0 and p_1. Similarly the last relation tells us that for $\ell \ne 0, 1$, r_ℓ can be expressed in terms of r_0 and r_1. From this it follows that $\mathcal{A}'_{B_n}$ is finitely generated. Using lemmas 3.3 and 3.4 to replace the first, second and sixth relations, assuming we have added a new generator b and relation $b = p_0q_3p_0^{-1}$, we arrive at the following theorem.

Theorem 3.9 *For every $n \ge 3$ the commutator subgroup $\mathcal{A}'_{B_n}$ of the Artin group $\mathcal{A}_{B_n}$ is a finitely generated group. Presentations for $\mathcal{A}'_{B_n}$, $n \ge 2$ are as follows:*
$\mathcal{A}'_{B_2}$ is a free group on countably many generators:

$$[a_2^\ell, a_1] \quad (\ell \in \mathbb{Z} \setminus \{0, \pm 1\}), \qquad [a_1^k a_2, a_1] \quad (k \in \mathbb{Z} \setminus \{0\}).$$

$\mathcal{A}'_{B_3}$ is a free group on four generators:

$$[a_1^{-1}, a_2^{-1}], \quad [a_3, a_2][a_1^{-1}, a_2^{-1}], \quad [a_1, a_2][a_1^{-1}, a_2^{-1}], \quad [a_1a_3, a_2][a_1^{-1}, a_2^{-1}].$$

$\mathcal{A}'_{B_4}$ is the group generated by

$$\begin{gathered}
p_k = a_1^k a_2 a_1^{-(k+1)} = [a_1^k, a_2][a_1^{-1}, a_2^{-1}], \quad (k \in \mathbb{Z})\\
q_\ell = a_4^\ell a_3 (a_1 a_4^\ell)^{-1} = [a_4^\ell, a_3][a_2^{-1}, a_3^{-1}][a_1^{-1}, a_2^{-1}], \quad (\ell \in \mathbb{Z}),
\end{gathered}$$

with defining relations

$$\begin{gathered}
p_{k+1}p_{k+2}^{-1}p_k^{-1} \quad (k \in \mathbb{Z}),\\
p_kq_\ell p_{k+2} = q_\ell p_{k+1} q_\ell \quad (k, \ell \in \mathbb{Z}),\\
q_\ell q_{\ell+1} = q_{\ell+1} q_{\ell+2} \quad (3 \le i \le n-3).
\end{gathered}$$

For $n \geq 5$ the group $\mathcal{A}'_{B_n}$ is generated by

$$p_0 = a_2 a_1^{-1}, \quad p_1 = a_1 a_2 a_1^{-2}, \quad q_3 = a_3 a_1^{-1}, \quad r_\ell = a_n^\ell a_{n-1}(a_1 a_n^\ell)^{-1} \quad (\ell \in \mathbb{Z}),$$
$$b = a_2 a_1^{-1} a_3 a_2^{-1}, \quad q_i = a_i a_1^{-1} \quad (4 \leq i \leq n-2),$$

with defining relations

$$p_0 q_j = q_j p_1, \quad p_1 q_j = q_j p_o^{-1} p_1 \quad (4 \leq j \leq n-2),$$
$$p_0 r_\ell = r_\ell p_1, \quad p_1 r_\ell = r_\ell p_0^{-1} p_1 \quad (\ell \in \mathbb{Z}),$$
$$q_i q_j = q_j q_i \quad (3 \leq i < j \leq n-2, \ |i = j| \geq 2),$$
$$q_i r_\ell = r_\ell q_i \quad (3 \leq i \leq n-3),$$
$$p_0 q_3 p_0^{-1} = b, \quad p_0 b p_0^{-1} = b^2 q_3^{-1} b,$$
$$p_1 q_3 p_1^{-1} = q_3^{-1} b, \quad p_1 b p_1^{-1} = (q_3^{-1} b)^3 q_3^{-2} b,$$
$$q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \quad (3 \leq i \leq n-3),$$
$$q_{n-2} r_\ell q_{n-2} = r_\ell q_{n-2} r_\ell \quad (\ell \in \mathbb{Z}),$$
$$r_\ell r_{\ell+1} r_{\ell+2}^{-1} r_{\ell+1}^{-1} \quad (\ell \in \mathbb{Z}),$$

□

Corollary 3.10 *For $n \geq 5$ the commutator subgroup $\mathcal{A}'_{B_n}$ of the Artin group of type B_n is finitely generated and perfect.*

Proof. Abelianizing the presentation of $\mathcal{A}'_{B_n}$ in the theorem results in a presentation of the trivial group. Hence $\mathcal{A}''_{B_n} = \mathcal{A}'_{B_n}$. □

3.3.4 Type D

The presentation of $\mathcal{A}_{D_n}$ is

$$\begin{aligned} \mathcal{A}_{D_n} = \langle a_1, ..., a_n : \quad & a_i a_j = a_j a_i \quad \text{for } 1 \leq i < j \leq n-1, |i-j| \geq 2, \\ & a_n a_j = a_j a_n \quad \text{for } j \neq n-2, \\ & a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad \text{for } 1 \leq i \leq n-2 \\ & a_{n-2} a_n a_{n-2} = a_n a_{n-2} a_n \rangle. \end{aligned}$$

As seen in figure 2 the graph $(D_n)_{odd}$ is connected. So by proposition 3.1

$$\mathcal{A}'_{D_n} = \{U \in \mathcal{A}_{D_n} : \deg(U) = 0\}.$$

The computation of the presentation of $\mathcal{A}'_{D_n}$ is similar to that of $\mathcal{A}'_{A_n}$, so we will not include it.

Theorem 3.11 *For every $n \geq 4$ the commutator subgroup $\mathcal{A}'_{D_n}$ of the Artin group $\mathcal{A}_{D_n}$ is a finitely presented group. $\mathcal{A}'_{D_4}$ is the group generated by*

$$p_0 = a_2 a_1^{-1}, \quad p_1 = a_1 a_2 a_1^{-2}, \quad q_3 = a_3 a_1^{-1},$$
$$q_4 = a_4 a_1^{-1}, \quad b = a_2 a_1^{-1} a_3 a_2^{-1}, c = a_2 a_1^{-1} a_4 a_2^{-1},$$

with defining relations

$$b = p_0 q_3 p_0^{-1}, \quad p_0 b p_0^{-1} = b^2 q_3^{-1} b,$$
$$p_1 q_3 p_1^{-1} = q_3^{-1} b, \quad p_1 b p_1^{-1} = (q_3^{-1} b)^3 q_3^{-2} b,$$
$$c = p_0 q_4 p_0^{-1}, \quad p_0 c p_0^{-1} = c^2 q_4^{-1} c,$$
$$p_1 q_4 p_1^{-1} = q_4^{-1} c, \quad p_1 c p_1^{-1} = (q_4^{-1} c)^3 q_4^{-2} c,$$
$$q_3 q_4 = q_4 q_3.$$

For $n \geq 5$ the group $\mathcal{A}'_{D_n}$ is generated by

$$p_0 = a_2 a_1^{-1}, \quad p_1 = a_1 a_2 a_1^{-2},$$
$$q_\ell = a_\ell a_1^{-1} \quad (3 \leq \ell \leq n), \quad b = a_2 a_1^{-1} a_3 a_2^{-1},$$

with defining relations

$$b = p_0 q_3 p_0^{-1}, \quad p_0 b p_0^{-1} = b^2 q_3^{-1} b,$$
$$p_1 q_3 p_1^{-1} = q_3^{-1} b, \quad p_1 b p_1^{-1} = (q_3^{-1} b)^3 q_3^{-2} b,$$
$$p_0 q_j = q_j p_1, \quad p_1 q_j = q_j p_0^{-1} p_1 \quad (4 \leq j \leq n),$$
$$q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \quad (3 \leq i \leq n-2),$$
$$q_n q_{n-2} q_n = q_{n-2} q_n q_{n-2},$$
$$q_i q_j = q_j q_i \quad (3 \leq i < j \leq n-1, |i-j| \geq 2),$$
$$q_n q_j = q_j q_n \quad (j \neq n-2).$$

□

Corollary 3.12 *For $n \geq 5$ the commutator subgroup $\mathcal{A}'_{D_n}$ of the Artin group of type D_n is finitely presented and perfect.* □

3.3.5 Type E

The presentation of $\mathcal{A}_{E_n}$, $n = 6, 7$, or 8, is

$$\begin{aligned}
\mathcal{A}_{E_n} = \langle a_1, ..., a_n : \quad & a_i a_j = a_j a_i \quad \text{for } 1 \leq i < j \leq n-1, |i-j| \geq 2, \\
& a_i a_n = a_n a_i \quad \text{for } i \neq 3, \\
& a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad \text{for } 1 \leq i \leq n-2 \\
& a_3 a_n a_3 = a_n a_3 a_n \rangle.
\end{aligned}$$

As seen in figure 2 the graph $(E_n)_{odd}$ is connected. So by proposition 3.1

$$\mathcal{A}'_{E_n} = \{U \in \mathcal{A}_{E_n} : \deg(U) = 0\}.$$

The computation of the presentation of $\mathcal{A}'_{E_n}$ is similar to that of $\mathcal{A}'_{A_n}$.

Theorem 3.13 *For $n = 6, 7,$ or 8 the commutator subgroup $\mathcal{A}'_{E_n}$ of the Artin group $\mathcal{A}_{E_n}$ is a finitely presented group. $\mathcal{A}'_{E_n}$ is the group generated by*

$$p_0 = a_2 a_1^{-1}, \quad p_1 = a_1 a_2 a_1^{-2}, \quad q_\ell = a_\ell a_1^{-1} \ \ (3 \le \ell \le n), \quad b = a_2 a_1^{-1} a_3 a_2^{-1},$$

with defining relations

$$\begin{gathered}
b = p_0 q_3 p_0^{-1}, \quad p_0 b p_0^{-1} = b^2 q_3^{-1} b, \\
p_1 q_3 p_1^{-1} = q_3^{-1} b, \quad p_1 b p_1^{-1} = (q_3^{-1} b)^3 q_3^{-2} b, \\
p_0 q_j = q_j p_1, \quad p_1 q_j = q_j p_0^{-1} p_1 \quad (4 \le j \le n), \\
q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \quad (3 \le i \le n-2), \\
q_n q_3 q_n = q_3 q_n q_3, \\
q_i q_j = q_j q_i \quad (3 \le i < j \le n-1, |i-j| \ge 2), \\
q_i q_n = q_n q_i \quad (4 \le i \le n-1).
\end{gathered}$$

□

Corollary 3.14 *For $n = 6, 7,$ or 8 the commutator subgroup $\mathcal{A}'_{E_n}$ of the Artin group of type E_n is finitely presented and perfect.* □

3.3.6 Type F

The presentation of $\mathcal{A}_{F_4}$ is

$$\begin{aligned}
\mathcal{A}_{F_4} = \langle a_1, a_2, a_3, a_4 : \qquad & a_i a_j = a_j a_i \quad \text{for } |i-j| \ge 2, \\
& a_1 a_2 a_1 = a_2 a_1 a_2, \\
& a_2 a_3 a_2 a_3 = a_3 a_2 a_3 a_2, \\
& a_3 a_4 a_3 = a_4 a_3 a_4 \rangle.
\end{aligned}$$

As seen in figure 2 the graph $(E_n)_{odd}$ has two components: Γ_1 and Γ_2, where Γ_1 denotes the component containing the vertices a_1, a_2, and Γ_2 the component containing the vertices a_3, a_4. Let $\deg_1$ and $\deg_2$ denote the associated degree maps, respectively, so from proposition 3.2

$$\mathcal{A}'_{F_4} = \{U \in \mathcal{A}_{F_4} : \deg_1(U) = 0 \text{ and } \deg_2(U) = 0\}.$$

By a computation similar to that of B_n we get the following.

Theorem 3.15 *The commutator subgroup $\mathcal{A}'_{F_4}$ of the Artin group of type F_4 is the group generated by*

$$\begin{gathered}
p_k = a_1^k a_2 a_1^{-(k+1)} = [a_1^k, a_2][a_1^{-1}, a_2^{-1}] \quad (k \in \mathbb{Z}), \\
q_\ell = a_4^\ell a_3 a_4^{-(\ell+1)} = [a_4^\ell, a_3][a_4^{-1}, a_3^{-1}] \quad (\ell \in \mathbb{Z}),
\end{gathered}$$

with defining relations

$$\begin{gathered}
p_k = p_{k+1} p_{k+2}^{-1} \ \ (k \in \mathbb{Z}), \quad q_\ell = q_{\ell+1} q_{\ell+2}^{-1} \ \ (\ell \in \mathbb{Z}), \\
p_k q_\ell p_{k+1} q_{\ell+1} = q_\ell p_k q_{\ell+1} p_{k+1} \ \ (k, \ell \in \mathbb{Z}).
\end{gathered}$$

The first two types of relations in the above presentation tell us that for $k \neq 0, 1$, p_k can be expressed in terms of p_0 and p_1, and similarly for q_ℓ. Thus $\mathcal{A}'_{F_4}$ is finitely generated. However, $\mathcal{A}'_{F_4}$ is not perfect since abelianizing the above presentation gives $\mathcal{A}'_{F_4}/\mathcal{A}''_{F_4} \simeq \mathbb{Z}^4$.

3.3.7 Type H

The presentation of $\mathcal{A}_{H_n}$, $n = 3$ or 4, is

$$\begin{aligned} \mathcal{A}_{H_n} = \langle a_1, ..., a_n : \quad & a_i a_j = a_j a_i \quad \text{for } |i-j| \geq 2, \\ & a_1 a_2 a_1 a_2 a_1 = a_2 a_1 a_2 a_1 a_2, \\ & a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad \text{for } 2 \leq i \leq n-1 \, \rangle. \end{aligned}$$

As seen in figure 2 the graph $(H_n)_{odd}$ is connected. So by proposition 3.1

$$\mathcal{A}'_{H_n} = \{U \in \mathcal{A}_{H_n} : \deg(U) = 0\}.$$

The computation of the presentation of $\mathcal{A}'_{H_n}$ is similar to that of $\mathcal{A}'_{A_n}$.

Theorem 3.16 *For $n = 3$ or 4 the commutator subgroup $\mathcal{A}'_{H_n}$ of the Artin group $\mathcal{A}_{H_n}$ is the group generated by*

$$p_k = a_1^k a_2 a_1^{-(k+1)} \quad (k \in \mathbb{Z}), \quad q_\ell = a_\ell a_1^{-\ell} \quad (3 \leq \ell \leq n),$$

with defining relations

$$\begin{gathered} p_k q_j = q_j p_{k+1} \quad (4 \leq j \leq n), \\ p_{k+1} p_{k+3} p_{k+4}^{-1} p_{k+2}^{-1} p_k^{-1} \quad (k \in \mathbb{Z}), \\ p_k q_3 p_{k+2} q_3^{-1} p_{k+1}^{-1} q_3^{-1} \\ q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \quad (3 \leq i \leq n-1). \end{gathered}$$

□

The second relation tells us that for $k \neq 0, 1, 2, 3$, p_k can be expressed in terms of p_0, p_1, p_2, and p_3. Thus, $\mathcal{A}'_{H_n}$ is finitely generated. Abelianizing the above presentation results in the trivial group. Thus, we have the following.

Corollary 3.17 *For $n = 3$ or 4 the commutator subgroup $\mathcal{A}'_{H_n}$ of the Artin group of type H_n is finitely generated and perfect.* □

3.3.8 Type I

The presentation of $I_2(m)$, $m \geq 5$, is

$$\mathcal{A}_{I_2(m)} = \langle a_1, a_2 : \langle a_1 a_2 \rangle^m = \langle a_2 a_1 \rangle^m \rangle.$$

In figure 2 the graph $(I_2(m))_{odd}$ is connected for m odd and disconnected for m even. Thus, different computations must be done for these two cases. We have the following.

Type Γ	finitely generated/presented	perfect
A_n	yes/yes	$n = 1, 2, 3 :$ no, $n \geq 4 :$ yes
B_n	$n = 2 :$ no, $n \geq 3 :$ yes / $n = 3 :$ yes, $n \geq 3 :$?	$n = 2, 3, 4 :$ no, $n \geq 5 :$ yes
D_n	yes/yes	$n = 4 :$ no, $n \geq 5 :$ yes
E_n	yes/yes	yes
F_4	yes/?	no
H_n	yes/?	yes
$I_2(m)$ (m even)	no/no	no
(m odd)	yes/yes	no

Table 2: Properties of the commutator subgroups

Theorem 3.18 *The commutator subgroup $\mathcal{A}'_{I_2(m)}$ of the Artin group of type $I_2(m)$, $m \geq 5$, is the free group generated by the $(m-1)$-generators*

$$a_1^k a_2 a_1^{-(k+1)} \quad (k \in \{0, 1, 2, \ldots, m-2\}),$$

when m is odd, and is the free group with countably many generators

$$[a_2^\ell, a_1] \quad (\ell \in \mathbb{Z} \setminus \{-(m/2-1)\}), \quad [a_1^j a_2^\ell, a_1] \quad (\ell \in \mathbb{Z}, \quad j = 1, 2, \ldots, m/2-3),$$
$$[a_1^{m/2-2} a_2^\ell, a_1] \quad (\ell \in \mathbb{Z} \setminus \{m/2-1\}), \quad [a_1^k a_2, a_1] \quad (k \in \mathbb{Z}).$$

when m is even.

3.3.9 Summary of results

Table 2 summarizes the results in this section. The question marks (?) in the table indicate that it is unknown whether the commutator subgroup is finitely presented. However, we do know that for these cases the group is finitely generated. If one finds more general relation equivalences along the lines of lemmas 3.3 and 3.4 then we may be able to show that these groups are indeed finitely presented.

4 Local indicability

4.1 Definitions and generalities

A group G is **indicable** if there exists a *nontrivial* homomorphism $G \longrightarrow \mathbb{Z}$ (called an indexing function). A group G is **locally indicable** if every nontrivial, finitely generated subgroup is indicable. Notice, finite groups cannot be indicable, so locally indicable groups must be torsion-free.

Local indicability was introduced by Higman [Hig40] in connection with the zero-divisor conjecture, which asserts that if G is a torsion-free group, then its integral group ring $\mathbb{Z}G$ has no zero divisors. Although still unsolved, the conjecture is true for groups which are locally indicable.

Every free group is locally indicable. Indeed, it is well known that every subgroup of a free group is itself free, and since free groups are clearly indicable the result follows.

Local indicability is clearly inherited by subgroups. The following simple theorem shows that the category of locally indicable groups is preseved under extensions.

Theorem 4.1 *If K, H and G are groups such that K and H are locally indicable and fit into a short exact sequence*

$$1 \longrightarrow K \xrightarrow{\ \phi\ } G \xrightarrow{\ \psi\ } H \longrightarrow 1,$$

then G is locally indicable.

Proof. Let $g_1, \ldots, g_n \in G$, and let $\langle g_1, \ldots, g_n \rangle$ denote the subgroup of G which they generate. If $\psi(\langle g_1, \ldots, g_n \rangle) \neq \{1\}$ then by the local indicability of H there exists a nontrivial homomorphism $f : \psi(\langle g_1, \ldots, g_n \rangle) \longrightarrow \mathbb{Z}$. Thus, the map

$$f \circ \psi : \langle g_1, \ldots, g_n \rangle \longrightarrow \mathbb{Z}$$

is nontrivial. Else, if $\psi(\langle g_1, \ldots, g_n \rangle) = \{1\}$ then $g_1, \ldots, g_n \in \ker\psi = \mathrm{Im}\phi$ (by exactness), so there exist $k_1, \ldots, k_n \in K$ such that $\phi(k_i) = g_i$, for all i. Since ϕ is one-to-one (short exact sequence) then $\phi : \langle k_1, \ldots, k_n \rangle \longrightarrow \langle g_1, \ldots, g_n \rangle$ is an isomorphism. By the local indicability of K there exists a nontrivial homomorphism $h : \langle k_1, \ldots, k_n \rangle \longrightarrow \mathbb{Z}$, therefore the map

$$h \circ \phi^{-1} : \langle g_1, \ldots, g_n \rangle \longrightarrow \mathbb{Z}$$

is nontrivial. $\square$

Corollary 4.2 *If G and H are locally indicable then so is $G \oplus H$.*

Proof. The sequence

$$1 \longrightarrow H \xrightarrow{\ \phi\ } G \oplus H \xrightarrow{\ \psi\ } G \longrightarrow 1$$

where $\phi(h) = (1, h)$ and $\psi(g, h) = g$ is exact, so the theorem applies. $\square$

If G and H are groups and $\phi : G \longrightarrow \mathrm{Aut}(H)$. The **semidirect product** of G and H is defined to be the set $H \times G$ with binary operation

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \cdot g_1 * h_2, g_1 g_2)$$

where $g * h$ denotes the action of G on H determined by ϕ, i.e. $g * h := \phi(g)(h) \in H$. This group is denoted by $H \rtimes_\phi G$.

Corollary 4.3 *If G and H are locally indicable then so is $H \rtimes_\phi G$.*

Proof. If $\psi : H \rtimes_\phi G \longrightarrow G$ denotes the map $(h, g) \longmapsto g$ then kerψ = H and the groups fit into the exact sequence

$$1 \longrightarrow H \xrightarrow{\text{incl.}} H \rtimes_\phi G \xrightarrow{\psi} G \longrightarrow 1$$

□

The following theorem of Brodskii [Bro80], [Bro84], which was discovered independently by Howie [How82], [How00], tells us that the class of torsion-free 1-relator groups lies inside the class of locally indicable groups. Also, for 1-relator groups: locally indicable ⇔ torsion free.

Theorem 4.4 *A torsion-free* 1*-relator group is locally indicable.*

To show a group is not locally indicable we need to show there exists a finitely generated subgroup in which the only homomorphism into $\mathbb{Z}$ is the trivial homomorphism.

Theorem 4.5 *If G contains a finitely generated perfect sugroup then G is not locally indicable.*

Proof. The image of a commutator $[a, b] := aba^{-1}b^{-1}$ under a homomorphism into $\mathbb{Z}$ is 0, thus the image of a perfect group is trivial. □

4.2 Local indicability of spherical Artin groups

Since spherical-type Artin groups are torsion-free, theorem 4.4 implies that the Artin groups of type A_2, B_2, and $I_2(m)$ ($m \geq 5$) are locally indicable. In this section we determine the local indicability of all[3] irreducible spherical-type Artin groups.

It is of interest to note that the discussion in section 3.2, in particular proposition 3.2, shows that an Artin group $\mathcal{A}_\Gamma$ and its commutator subgroup $\mathcal{A}'_\Gamma$ fit into a short exact sequence:

$$1 \longrightarrow \mathcal{A}'_\Gamma \longrightarrow \mathcal{A}_\Gamma \xrightarrow{\deg_\Gamma} \mathbb{Z}^m \longrightarrow 1,$$

where m is the number of connected components in Γ_{odd}, and $\deg_\Gamma$ is the degree map defined in 3.2, which can be identified with the abelianization map. Thus, the local indicability of an Artin group $\mathcal{A}_\Gamma$ is completely determined by the local indicability of its commutator subgroup $\mathcal{A}'_\Gamma$ (by theorem 4.1). In other words,

$$\mathcal{A}_\Gamma \text{ is locally indicable} \Longleftrightarrow \mathcal{A}'_\Gamma \text{ is locally indicable.}$$

This gives another proof that the Artin groups of type A_2, B_2, and $I_2(m)$ ($m \geq 5$) are locally indicable, since their corresponding commutator subgroups are free groups, as already shown.

[3]with the exception of type F_4 which at this time remains undetermined.

4.2.1 Type A

A_{A_1} is clearly locally indicable since $A_{A_1} \simeq \mathbb{Z}$, and, as noted above, $\mathcal{A}_{A_2}$ is also locally indicable.

For $\mathcal{A}_{A_3}$, theorem 3.8 tells us $\mathcal{A}'_{A_3}$ is the semidirect product of two free groups, thus $\mathcal{A}'_{A_3}$ is locally indicable. It follows from our remarks above that $\mathcal{A}_{A_3}$ is also locally indicable.

As for $\mathcal{A}_{A_n}$, $n \geq 4$, corollary 3.7 and theorem 4.5 imply that $\mathcal{A}_{A_n}$ is not locally indicable.

Thus, we have the following theorem.

Theorem 4.6 $\mathcal{A}_{A_n}$ *is locally indicable if and only if* $n = 1, 2,$ *or* 3.

4.2.2 Type B

We saw above $\mathcal{A}_{B_2}$ is locally indicable. For $n = 3$ and 4 we argue as follows.

Let $\mathcal{P}^{n+1}_{n+1}$ denote the $(n+1)$-pure braids in $\mathfrak{B}_{n+1} = \mathcal{A}_{A_n}$, that is the braids which only permute the first n-strings. Letting $b_1, \ldots, b_n$ denote the generators of $\mathcal{A}_{B_n}$ a theorem of Crisp [Cri99] states

Theorem 4.7 *The map*

$$\phi : \mathcal{A}_{B_n} \longrightarrow \mathcal{A}_{A_n}$$

defined by

$$b_i \longmapsto a_i, \quad b_n \longmapsto a_n^2$$

is an injective homomorphism onto $\mathcal{P}^{n+1}_{n+1}$. *That is,* $\mathcal{A}_{B_n} \simeq \mathcal{P}^{n+1}_{n+1} < \mathfrak{B}_{n+1} = \mathcal{A}_{A_n}$.

By "forgetting the n^{th}-strand" we get a homomorphism $f : \mathcal{P}^{n+1}_{n+1} \longrightarrow \mathfrak{B}_n$ which fits into the short exact sequence

$$1 \longrightarrow K \longrightarrow \mathcal{P}^{n+1}_{n+1} \xrightarrow{\ f\ } \mathfrak{B}_n \longrightarrow 1,$$

where $K = \mathrm{ker\ f} = \{\beta \in \mathcal{P}^{\mathrm{n+1}}_{\mathrm{n+1}} : \text{the first } n \text{ strings of } \beta \text{ are trivial}\}$. It is known that $K \simeq F_n$, the free group of rank n. Since F_n is locally indicable and $\mathfrak{B}_n$ $(n = 3, 4)$ is locally indicable then so is $\mathcal{A}_{B_n}$, for $n = 3, 4$. Futhermore, the above exact sequence is actually a split exact sequence so $\mathcal{A}_{B_n} \simeq \mathcal{P}^{n+1}_{n+1} \simeq F_n \rtimes \mathfrak{B}_n$.

As for $\mathcal{A}_{B_n}$, $n \geq 5$, corollary 3.10 and theorem 4.5 imply that $\mathcal{A}_{B_n}$ is not locally indicable, for $n \geq 5$.

Thus, we have the following theorem.

Theorem 4.8 $\mathcal{A}_{B_n}$ *is locally indicable if and only if* $n \leq 4$.

4.2.3 Type D

It follows corollary 3.12 and 4.5 that $\mathcal{A}_{D_n}$ is not locally indicable for $n \geq 5$. As for $\mathcal{A}_{D_4}$, we will show it is locally indicable as follows.

A theorem of Crisp and Paris [CP02] says:

Theorem 4.9 *Let F_{n-1} denote the free group of rank $n-1$. There is an action $\rho : \mathcal{A}_{A_{n-1}} \longrightarrow \mathrm{Aut}(F_{n-1})$ such that $\mathcal{A}_{D_n} \simeq F_{n-1} \rtimes \mathcal{A}_{A_{n-1}}$ and ρ is faithful.*

Since $\mathcal{A}_{A_3}$ and F_3 are locally indicable, then so is $\mathcal{A}_{D_4}$. Thus, we have the following theorem.

Theorem 4.10 *$\mathcal{A}_{D_n}$ is locally indicable if and only if $n = 4$.*

4.2.4 Type E

Since the commutator subgroups of $\mathcal{A}_{E_n}$, $n = 6, 7, 8$, are finitely generated and perfect (corollary 3.14) then $\mathcal{A}_{E_n}$ is not locally indicable.

4.2.5 Type F

Unfortunately, we have yet to determine the local indicability of the Artin group $\mathcal{A}_{F_4}$.

4.2.6 Type H

Since the commutator subgroups of $\mathcal{A}_{H_n}$, $n = 3, 4$, are finitely generated and perfect (corollary 3.17) then $\mathcal{A}_{H_n}$ is not locally indicable.

4.2.7 Type I

As noted above, since the commutator subgroup $\mathcal{A}'_{I_2(m)}$ of $\mathcal{A}_{I_2(m)}$ ($m \geq 5$) is a free group (theorem 3.18) then $\mathcal{A}'_{I_2(m)}$ is locally indicable and therefore so is $\mathcal{A}_{I_2(m)}$. One could also apply theorem 4.4 to conclude the same result.

5 Open questions: Orderability

In this section we discuss the connection between the theory of orderable groups and the theory of locally indicable groups. Then we discuss the current state of the orderability of the irreducible spherical-type Artin groups.

5.1 Orderable Groups

A group or monoid G is **right-orderable** if there exists a strict linear ordering $<$ of its elements which is right-invariant: $g < h$ implies $gk < hk$ for all g, h, k in G. If there is an ordering of G which is invariant under multiplication on both sides, we say that G is **orderable** or for emphasis **bi-orderable** .

Theorem 5.1 *G is right-orderable if and only if there exists a subset $\mathcal{P} \subset G$ such that:*

$$\mathcal{P} \cdot \mathcal{P} \subset \mathcal{P} \text{ (subsemigroup)},$$
$$G \setminus \{1\} = \mathcal{P} \sqcup \mathcal{P}^{-1}.$$

Proof. Given $\mathcal{P}$ define $<$ by: $g < h$ iff $hg^{-1} \in \mathcal{P}$. Given $<$ take $\mathcal{P} = \{g \in G : 1 < g\}$. □

The ordering is a bi-ordering if and only if $\mathcal{P}$ exists as above and also

$$g\mathcal{P}g^{-1} \subset \mathcal{P}, \quad \forall g \in G.$$

The set $\mathcal{P} \subset G$ in the previous theorem is called the **positive cone** with respect to the ordering $<$.

The theory of orderable groups is well over a hundred years old. For a general exposition on the theory of orderable groups see [MR77] or [KK74]. We will list just a few properties of orderable groups.

A group is right-orderable if and only if it is left-orderable (by a possibly different ordering. The class of right-orderable groups is closed under: subgroups, direct products, free products, semidirect products, and extension. The class of orderable groups is closed under: subgroups, direct products, free products, but not necessarily extensions. Both right-orderability and bi-orderability are local properties: a group has the property if and only if every finitely-generated subgroup has it.

Knowing a group is right-orderable or bi-orderable provides useful information about the internal structure of the group. For example, if G is right-orderable then it must be torsion-free: for $1 < g$ implies $g < g^2 < g^3 < \cdots < g^n < \cdots$. Moreover, if G is bi-orderable then G has no **generalised torsion** (products of conjugates of a nontrivial element being trivial), G has unique roots: $g^n = h^n \Rightarrow g = h$, and if $[g^n, h] = 1$ in G then $[g, h] = 1$. Further consequences of orderablility are as follows. For any group G, let $\mathbb{Z}G$ denote the **group ring** of formal linear combinations $n_1g_1 + \cdots n_kg_k$.

Theorem 5.2 *If G is right-orderable, then $\mathbb{Z}G$ has no zero divisors.* □

Theorem 5.3 *(Malcev, Neumann) If G is bi-orderable, then $\mathbb{Z}G$ embeds in a division ring.* □

Theorem 5.4 *(LaGrange, Rhemtulla) If G is right orderable and H is any group, then $\mathbb{Z}G \simeq \mathbb{Z}H$ implies $G \simeq H$.* □

The following theorems give the connection between orderable groups and locally indicable groups (see [RR02]). The first is due to Levi [Lev43], and the second due to Burns and Hale [BH72] .

Theorem 5.5 *Every bi-orderable group is locally indicable.* □

The Artin group of type A_2, for example, shows the converse does not hold.

Theorem 5.6 *Every locally indicable group is right-orderable.* □

Bergman [Ber91] was the first to publish examples demonstrating that the converse to theorem 5.6 is false. Note that the Artin groups of type $A_n, n \geq 4$ are also examples of right-orderable groups which are not locally indicable.

One final connection between local indicability and right-orderability was given by Rhemtulla and Rolfsen [RR02].

Theorem 5.7 *(Rhemtulla, Rolfsen) Suppose $(G, <)$ is right-ordered and there is a finite-index subgroup H of G such that $(H, <)$ is a bi-ordered group. Then G is locally indicable.*

An application of this theorem is as follows. It is known that the braid groups $\mathfrak{B}_n = \mathcal{A}_{A_{n-1}}$ are right orderable [DDRW02] and that the pure braids $\mathcal{P}_n$ are bi-orderable [KR03]. However, theorem 4.6 tells us that $\mathfrak{B}_n$ is not locally indicable for $n \geq 5$ therefore, by theorem 5.7, the bi-ordering on $\mathcal{P}_n$ and the right-ordering on $\mathfrak{B}_n$ are incompatible for $n \geq 5$.

5.2 Ordering spherical Artin groups

The first proof the that braid groups $\mathfrak{B}_n$ enjoy a right-invariant total ordering was given in [Deh92], [Deh94]. Since then several quite different approaches have been applied to understand this phenomenon.[4] The following theorems summarize the state of our knowledge regarding orderability of the spherical Artin groups.

Theorem 5.8 *The Artin groups of type $A_n (n \geq 2), B, D, E, F, H$ and I are not bi-orderable.*

Proof. The Artin group of type A_2 is not biorderable, as it does not have unique roots: $aba = bab$ implies that $(ab)^3 = (ba)^3$, whereas $ab \neq ba$ (their images in the Coxeter group are distinct). For similar reasons, the Artin groups of type B_2 and $I_2(m)$ are not biorderable. The others listed all contain a type A_2 Artin group as a (parabolic) subgroup, and therefore cannot be bi-ordered. □

In other words, only the simplest irreducible spherical Artin group $\mathcal{A}_{A_1} \cong \mathbb{Z}$ can be bi-ordered. On the other hand, many of them (perhaps all those of spherical type) have a right-invariant ordering.

Theorem 5.9 *The Artin groups of type A, B, D and I are right-orderable.*

Proof. Type A are the braid groups, shown to be right-orderable by Dehornoy. Since the Artin groups of type B and I embed in type A, they are also right-orderable (alternatively, type I are right-orderable because they are locally indicable). Finally, theorem 4.9 and the fact that right-orderability is closed under extensions, implies the right-orderability of type D Artin groups. □

However, we do not know whether the remaining irreducible spherical-type Artin groups are right-orderable. One approach is to reduce the problem to showing that the positive Artin monoid is right-orderable. If Γ is a Coxeter graph, we define the corresponding Artin monoid $\mathcal{A}_\Gamma^+$ to be the monoid with the

[4]For a comprehensive look at this problem and all the different approaches used to understand it see the book [DDRW02].

same presentation as the Artin group, but interpreted as a monoid presentation. That is,

$$\mathcal{A}_\Gamma^+ = \langle a \in \Sigma : \langle ab \rangle^{m_{ab}} = \langle ba \rangle^{m_{ab}} \text{ if } m_{ab} < \infty \rangle^+.$$

It is known [Par02] that $\mathcal{A}_\Gamma^+$ injects in $\mathcal{A}_\Gamma$ as the submonoid of words in the canonical generators with no negative exponents.

5.2.1 Ordering the Monoid is Sufficient

We will show that for a Coxeter graph Γ of spherical type the Artin group $\mathcal{A}_\Gamma$ is right-orderable (resp. bi-orderable) if and only if the Artin monoid $\mathcal{A}_\Gamma^+$ is right-orderable (resp. bi-orderable). One direction is of course trivial.

Let $\mathcal{A}_\Gamma$ be an Artin group of spherical-type. Brieskorn and Saito [BS72], generalizing a result of Garside, have shown:

For each $U \in \mathcal{A}_\Gamma$ there exist $U_1, U_2 \in \mathcal{A}_\Gamma^+$, where U_2 is central in $\mathcal{A}_\Gamma$, such that

$$U = U_1 U_2^{-1}.$$

All decompositions of elements of $\mathcal{A}_\Gamma$ in this section are assumed to be of this form.

Suppose $\mathcal{A}_\Gamma^+$ is right-orderable, let $<^+$ be such a right-invariant linear ordering. We wish to prove that $\mathcal{A}_\Gamma$ is right-orderable.

The following lemma indicates how we should extend the ordering on the monoid to the entire group.

Lemma 5.10 *If $U \in \mathcal{A}_\Gamma$ has two decompositions;*

$$U = U_1 U_2^{-1} = \overline{U}_1 \overline{U}_2^{-1},$$

where U_i, $\overline{U}_i \in \mathcal{A}_\Gamma^+$ and U_2, $\overline{U}_2$ central in $\mathcal{A}_\Gamma$, then

$$U_1 <^+ U_2 \Longleftrightarrow \overline{U}_1 <^+ \overline{U}_2.$$

Proof. $U = U_1 U_2^{-1} = \overline{U}_1 \overline{U}_2^{-1}$ implies $U_1 \overline{U}_2 =_p \overline{U}_1 U_2$, ($=_p$ means equal in the monoid) since $U_2, \overline{U}_2$ central and $\mathcal{A}_\Gamma^+$ canonically injects in $\mathcal{A}_\Gamma$.
If $U_1 <^+ U_2$ then

$$\begin{aligned}
&\Rightarrow U_1 \overline{U}_2 <^+ U_2 \overline{U}_2 && \text{since } <^+ \text{ right-invariant,}\\
&\Rightarrow U_1 \overline{U}_2 <^+ \overline{U}_2 U_2 && \text{since } U_2 \text{ central,}\\
&\Rightarrow \overline{U}_1 U_2 <^+ \overline{U}_2 U_2 && \text{since } U_1 \overline{U}_2 =_p \overline{U}_1 U_2,\\
&\Rightarrow \overline{U}_1 <^+ \overline{U}_2,
\end{aligned}$$

where the last implication follows from the fact that if $\overline{U}_2 \leq^+ \overline{U}_1$ then either: (i) $\overline{U}_2 = \overline{U}_1$, in which case $U = 1$ and so $U_1 = U_2$, a contradiction, or (ii) $\overline{U}_2 <^+ \overline{U}_1$, in which case $\overline{U}_2 U_2 <^+ \overline{U}_1 U_2$. Again, a contradiction.

The reverse implication follows by symmetry. □

This lemma shows that the following set is well defined:

$$\mathcal{P} = \{U \in \mathcal{A}_\Gamma :\ U \text{ has decomposition } U = U_1 U_2^{-1} \text{ where } U_2 <^+ U_1\}.$$

It is an easy exercise to check that $\mathcal{P}$ is a positive cone in $\mathcal{A}_\Gamma$ which contains $\mathcal{P}^+$: the positive cone in $\mathcal{A}_\Gamma^+$ with respect to the order $<^+$. Thus, the right-invariant order $<^+$ on $\mathcal{A}_\Gamma^+$ extends to a right-invariant order $<$ on $\mathcal{A}_\Gamma$, and we have shown the following.

Theorem 5.11 *If Γ is a spherical-type Coxeter graph, then the Artin group $\mathcal{A}_\Gamma$ is right-orderable if and only if the corresponding Artin monoid $\mathcal{A}_\Gamma^+$ is right-orderable.* □

5.2.2 Reduction to type E_8

Table 1 shows that every irreducible spherical-type Artin group injects into one of type A, D, or E. According to theorem 5.9, the Artin groups of type A and D are right-orderable. The Artin group of types E_6 and E_7 naturally live inside $\mathcal{A}_{E_8}$, so it suffices to show $\mathcal{A}_{E_8}$ is right-orderable, to conclude that all Artin groups are right-orderable. At this point in time it is unknown whether $\mathcal{A}_{E_8}$ is right-orderable. As section 5.2.1 indicates it is enough to decide whether the Artin monoid $\mathcal{A}_{E_8}^+$ is right-orderable.

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